

Katětov functors

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Abstract

We develop a theory of *Katětov functors* which provide a uniform way of constructing Fraïssé limits. Among applications, we present short proofs and improvements of several recent results on the structure of the group of automorphisms and the semigroup of endomorphisms of some Fraïssé limits.

Keywords: Katětov functor, amalgamation, Fraïssé limit.

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Contents

1	Introduction	2
1.1	The setup	3
2	Katětov functors	6
2.1	Examples	10
2.2	Sufficient conditions for the existence of Katětov functors	13
3	Katětov construction	16
3.1	Characterizations of the existence of a Katětov functor	20
3.2	Automorphism groups and endomorphism monoids	23
4	Semigroup Bergman property	26
5	Appendix: the original Katětov construction	35
5.1	Conclusion	38

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1 Introduction

The theory of Fraïssé limits has a long history, inspired by Cantor's theorem saying that the set of rational numbers is the unique, up to isomorphisms, countable linearly ordered set without end-points and such that between any two points there is another one. In the fifties of the last century Roland Fraïssé realized that the ideas behind Cantor's theorem are much more general, and he developed his theory of limits. Namely, given a class \mathcal{A} of finitely generated first-order structures with certain natural properties, there exists a unique countably generated structure L (called the *Fraïssé limit* of \mathcal{A}) containing isomorphic copies of all structures from \mathcal{A} and having a very strong homogeneity property, namely, every isomorphism between finitely generated substructures of L extends to an automorphism of L . Fraïssé theory can now be called *classical* and it is part of almost every textbook in model theory. Independently of Fraïssé, around thirty years earlier, Urysohn [19] constructed a universal separable complete metric space \mathbb{U} which has the same homogeneity property as Fraïssé limits: every finite isometry extends to a bijective isometry of \mathbb{U} . Around the eighties of the last century, merely sixty years after Urysohn's work, Katětov [12] found a uniform way of extending metric spaces, leading to a new simple construction of the Urysohn space \mathbb{U} . It turns out that Katětov's construction is functorial, namely, it can be extended to all nonexpansive mappings between metric spaces.

We address the question when a functorial way of constructing a Fraïssé limit exists. Namely, we define the concept of a *Katětov functor* capturing simple extensions of finitely generated structures, whose infinite power gives the Fraïssé limit. The existence of a Katětov functor implies directly that the automorphism group of the Fraïssé limit is universal for the class of all automorphism groups of countably generated structures from the given Fraïssé class. Papers [8] and [3] discuss some of the issues addressed in this note, without realizing that what one deals with are actually functorial constructions.

As we have mentioned above, our principal motivation comes from Katětov's construction of the Urysohn space [12], which we briefly recall here in case of the rational Urysohn space. Let X be a metric space with rational distances. A *Katětov function over X* is every function $\alpha : X \rightarrow \mathbb{Q}$ such that

$$|\alpha(x) - \alpha(y)| \leq d(x, y) \leq \alpha(x) + \alpha(y)$$

for all $x, y \in X$. Let $K(X)$ be the set of all Katětov functions over X . The sup metric turns $K(X)$ into a metric space. There is a natural isometric embedding $X \hookrightarrow K(X)$ which takes $a \in X$ to $d(a, \cdot) \in K(X)$. Hence we get a chain of embeddings

$$X \hookrightarrow K(X) \hookrightarrow K^2(X) \hookrightarrow K^3(X) \hookrightarrow \dots$$

whose colimit is easily seen to be the rational Urysohn space.

It was observed by several authors (see, e.g., [2], [20]) that the construction K is actually functorial with respect to embeddings. Our principal observation is that more is true: if \mathcal{A} is the category of all finite metric spaces with rational distances and nonexpansive mappings, and \mathcal{C} is the category of all countable metric spaces with rational distances and nonexpansive mappings, then K can be turned into a functor from \mathcal{A} to \mathcal{C} . We present the details in the last section.

The paper is organized as follows. Section 2 contains the main concept of a Katětov functor, its basic properties, examples, and a discussion of sufficient conditions for its existence. We prove, in particular, that a Katětov functor exists if embeddings have pushouts in the category of all homomorphisms. In Section 3 we show how iterations of a Katětov functor lead to Fraïssé limits. It turns out that the Fraïssé limit can be viewed as a fixed point of the countable infinite power of a Katětov functor and all orbits of this functor “tend” to the Fraïssé limit, resembling the Banach contraction principle. Section 4 deals with the semigroup Bergman property. We prove that in the presence of a Katětov functor, under some mild additional assumptions the endomorphism monoid $\text{End}(L)$ of the Fraïssé limit L is strongly distorted and its Sierpiński rank is at most five. Applying a result from [16], we conclude that if $\text{End}(L)$ is not finitely generated, then it has the Bergman property. This extends a recent result of Dolinka [6]. The last Section 5 is an appendix containing description of the original Katětov functor on metric spaces with nonexpansive mappings.

1.1 The setup

Let $\Delta = \mathcal{R} \cup \mathcal{F} \cup \mathcal{C}$ be a first-order language, where \mathcal{R} is a set of relational symbols, \mathcal{F} a set of functional symbols, and \mathcal{C} a set of constant symbols. We say that Δ is a *purely relational language* if $\mathcal{F} = \mathcal{C} = \emptyset$. For a Δ -structure A and $X \subseteq A$, by $\langle X \rangle_A$ we denote the substructure of A generated by X . We say that A is *finitely generated* if $A = \langle X \rangle_A$ for some finite $X \subseteq A$. The fact that A is a substructure of B will be denoted by $A \leq B$.

Let \mathcal{C} be a category of Δ -structures. A *chain in \mathcal{C}* is a sequence of objects and embeddings of the form $C_1 \hookrightarrow C_2 \hookrightarrow C_3 \hookrightarrow \dots$. Note that although there may be other kinds of morphisms in \mathcal{C} , a chain always consists of objects and embeddings. We shall say that L is a *standard colimit* of the chain $C_1 \hookrightarrow C_2 \hookrightarrow \dots$ if it is a colimit of this chain in the usual sense and moreover, after forgetting the structure L is still a colimit in the category of sets. In other words, if the embeddings are inclusions, that is, $C_1 \leq C_2 \leq \dots$ then a standard colimit is $L = \bigcup_{n \in \mathbb{N}} C_n$ with an appropriate Δ -structure making it a colimit in \mathcal{C} . We shall say that \mathcal{C} has *standard colimits of chains* if every chain in \mathcal{C} has a standard colimit in \mathcal{C} . Given $C \in \text{Ob}(\mathcal{C})$, let $\text{Aut}(C)$ denote the permutation group consisting of all automorphisms of C , and let $\text{End}(C)$ denote the transformation monoid consisting of all \mathcal{C} -morphisms $C \rightarrow C$. It may be the case that $\text{End}(C)$ consists of all embeddings of C into C (if \mathcal{C}

consists of embeddings only). We shall sometimes write $\text{End}_{\mathcal{C}}(C)$ instead of $\text{End}(C)$ in order to emphasize that we consider \mathcal{C} -morphisms only. Let $\text{age}(C)$ denote the class of all finitely generated objects that embed into C . We say that \mathcal{A} has the *joint-embedding property* (briefly: (JEP)) if every two structures in \mathcal{A} embed into a common structure in \mathcal{A} .

Standing assumption. Throughout the paper we assume the following. Let Δ be a first-order language, let \mathcal{C} be a category of countably generated Δ -structures and some appropriately chosen class of morphisms that includes all embeddings (and hence all isomorphisms). Let \mathcal{A} be the full subcategory of \mathcal{C} spanned by all finitely generated structures in \mathcal{C} . In particular, \mathcal{A} is *hereditary* in the sense that given $A \in \text{Ob}(\mathcal{A})$, every finitely generated substructure¹ of A is an object of \mathcal{A} .

We assume that the following holds:

- \mathcal{C} has standard colimits of chains;
- every $C \in \text{Ob}(\mathcal{C})$ is a colimit of some chain $A_1 \hookrightarrow A_2 \hookrightarrow \dots$ in \mathcal{A} ;
- \mathcal{A} has only countably many isomorphism types; and
- \mathcal{A} has the joint embedding property (JEP).

We say that $C \in \text{Ob}(\mathcal{C})$ is a *one-point extension* of $B \in \text{Ob}(\mathcal{C})$ if there is an embedding $j : B \hookrightarrow C$ and an $x \in C \setminus j(B)$ such that $C = \langle j(B) \cup \{x\} \rangle_C$. In that case we write $j : B \hookrightarrow C$ or simply $B \hookrightarrow C$.

The following lemmas are immediate consequences of the fact that \mathcal{C} is a category of Δ -structures and the fact that \mathcal{A} is spanned by finitely generated objects in \mathcal{C} .

Lemma 1.1 (Reachability) (a) For all $A, B \in \text{Ob}(\mathcal{A})$ and an embedding $A \hookrightarrow B$ which is not an isomorphism, there exist an $n \in \mathbb{N}$ and $A_1, \dots, A_n \in \text{Ob}(\mathcal{A})$ such that $A \hookrightarrow A_1 \hookrightarrow A_2 \hookrightarrow \dots \hookrightarrow A_n = B$.

(b) For all $C, D \in \text{Ob}(\mathcal{C})$ and an embedding $f : C \hookrightarrow D$ which is not an isomorphism, there exist $C_1, C_2 \dots \in \text{Ob}(\mathcal{C})$ such that

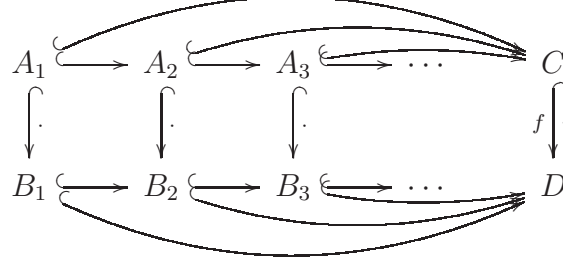
$$\begin{array}{ccccccc}
 C & \hookrightarrow & C_1 & \hookrightarrow & C_2 & \hookrightarrow & \dots \\
 & \searrow & & \searrow & \downarrow & \nearrow & \\
 & & & & D & &
 \end{array}$$

(The arrow from C to D is labeled f)

is a colimit diagram in \mathcal{C} .

¹Recall that substructures of finitely generated structures may not be finitely generated. For example, the free group with 2 generators has a subgroup isomorphic to the free group with infinitely many generators.

Lemma 1.2 Let $C, D \in \text{Ob}(\mathcal{C})$ be structures such that $f : C \hookrightarrow D$ and let $A_1 \hookrightarrow A_2 \hookrightarrow \dots$ be a chain in \mathcal{A} whose colimit is C . Then there exists a chain $B_1 \hookrightarrow B_2 \hookrightarrow \dots$ in \mathcal{A} whose colimit is D and the following diagram commutes



where the curvy arrows are canonical embeddings into the colimits.

Proof. Without loss of generality we can assume that $C \leq D$, and that $A_1 \leq A_2 \leq \dots \leq C$, so that $C = \bigcup_{i \in \mathbb{N}} A_i$. Since D is a one-point extension of C , there exists an $x \in D \setminus C$ such that $D = \langle C \cup \{x\} \rangle_D$. Put $B_i = \langle A_i \cup \{x\} \rangle_D$. \square

The next lemma is rather obvious, as we assume that colimits are standard.

Lemma 1.3 (Factoring through the colimit of a chain) Let

$$C_1 \hookrightarrow C_2 \hookrightarrow \dots$$

be a chain in \mathcal{C} and let L be its colimit with the canonical embeddings $\iota_k : C_k \hookrightarrow L$. Then for every $A \in \text{Ob}(\mathcal{A})$ and every morphism $f : A \rightarrow L$ there is an $n \in \mathbb{N}$ and a morphism $g : A \rightarrow C_n$ such that $f \circ g = \iota_n$. Moreover, if f is an embedding, then so is g .

$$\begin{array}{ccc} & C_n & \\ g \nearrow & \downarrow \iota_n & \\ A & \xrightarrow{f} & L \end{array}$$

Lemma 1.4 For every $C \in \text{Ob}(\mathcal{C})$ we have that $\text{age}(C) \subseteq \text{Ob}(\mathcal{A})$.

Proof. Take any $C \in \text{Ob}(\mathcal{C})$, and let $A_1 \hookrightarrow A_2 \hookrightarrow \dots$ be a chain in \mathcal{A} whose colimit is C . Take any $B \in \text{age}(C)$. Then $B \hookrightarrow C$, so by Lemma 1.3 there is an $n \in \mathbb{N}$ and an embedding $g : B \hookrightarrow A_n$ such that

$$\begin{array}{ccc} & A_n & \\ g \nearrow & \downarrow & \\ B & \hookrightarrow & C \end{array}$$

Therefore, $B \hookrightarrow A_n \in \text{Ob}(\mathcal{A})$, so the assumption that \mathcal{A} is hereditary yields $B \in \text{Ob}(\mathcal{A})$. \square

2 Katětov functors

Definition 2.1 A functor $K^0 : \mathcal{A} \rightarrow \mathcal{C}$ is a *Katětov functor* if:

- K^0 preserves embeddings, that is, if $f : A \rightarrow B$ is an embedding in \mathcal{A} , then $K^0(f) : K^0(A) \rightarrow K^0(B)$ is an embedding in \mathcal{C} ; and
- there is a natural transformation $\eta^0 : \text{ID} \rightarrow K^0$ such that for every one-point extension $A \hookrightarrow B$ where $A, B \in \text{Ob}(\mathcal{A})$, there is an embedding $g : B \hookrightarrow K^0(A)$ satisfying

$$\begin{array}{ccc} A & \xrightarrow{\eta_A^0} & K^0(A) \\ \downarrow & \nearrow g & \\ B & & \end{array} \quad (1)$$

Theorem 2.2 If there exists a Katětov functor $K^0 : \mathcal{A} \rightarrow \mathcal{C}$ then there is a functor $K : \mathcal{C} \rightarrow \mathcal{C}$ such that:

- K is an extension of K^0 (that is, K and K^0 coincide on \mathcal{A});
- there is a natural transformation $\eta : \text{ID} \rightarrow K$ which is an extension of η^0 (that is, $\eta_A = \eta_A^0$ whenever $A \in \text{Ob}(\mathcal{A})$);
- K preserves embeddings.

Proof. The obvious candidate for K is the left Kan extension of K^0 along the inclusion functor $E : \mathcal{A} \rightarrow \mathcal{C}$ (which acts identically on both objects and morphisms of \mathcal{A}). To show that such an extension exists it suffices to show that the diagram $(E \downarrow \mathcal{C}) \xrightarrow{\Pi} \mathcal{A} \xrightarrow{K^0} \mathcal{C}$ has a colimit in \mathcal{C} for every $C \in \text{Ob}(\mathcal{C})$, where Π is the projection functor from the comma category $(E \downarrow \mathcal{C})$ to \mathcal{A} which takes an object $(A, h : A \rightarrow C)$ of the comma category to its first coordinate A , and acts on morphisms accordingly [15].

Take any $C \in \text{Ob}(\mathcal{C})$ and let $A_1^C \hookrightarrow A_2^C \hookrightarrow \dots$ be a chain in \mathcal{A} whose colimit is C . Let $\iota_n^C : A_n^C \hookrightarrow C$ be the canonical embeddings. Recall that for every $B \in \text{Ob}(\mathcal{A})$ and every morphism $f : B \rightarrow C$ there is an n and a morphism $f_n : B \rightarrow A_n^C$ such that $\iota_n^C \circ f_n = f$ (Lemma 1.3):

$$\begin{array}{ccc} A_n^C & \hookrightarrow & A_{n+1}^C \\ f_n \uparrow & \searrow \iota_n^C & \downarrow \iota_{n+1}^C \\ B & \xrightarrow{f} & C \end{array}$$

The diagram $(E \downarrow C) \xrightarrow{\Pi} \mathcal{A} \xrightarrow{K^0} \mathcal{C}$ then takes the form

$$\begin{array}{ccc} K^0(A_n^C) & \hookrightarrow & K^0(A_{n+1}^C) \\ K^0(f_n) \uparrow & & \\ K^0(B) & & \end{array}$$

Let D be the colimit of the chain $K^0(A_1^C) \hookrightarrow K^0(A_2^C) \hookrightarrow \dots$ with the canonical embeddings $\iota_n^D : K^0(A_n^C) \hookrightarrow D$. For each $f : B \rightarrow C$ in \mathcal{A} let $f' = \iota_n^D \circ K_0(f_n) : K^0(B) \rightarrow D$:

$$\begin{array}{ccc} K^0(A_n^C) & \hookrightarrow & K^0(A_{n+1}^C) \\ K^0(f_n) \uparrow & \searrow \iota_n^D & \downarrow \iota_{n+1}^D \\ K^0(B) & \xrightarrow{f'} & D \end{array}$$

Then it is easy to show that D is the colimit of the diagram $(E \downarrow C) \xrightarrow{\Pi} \mathcal{A} \xrightarrow{K^0} \mathcal{C}$ in \mathcal{C} . Therefore, K^0 has the left Kan extension K along E .

Let us show that K preserves embeddings. Take any embedding $f : C \hookrightarrow D$ in \mathcal{C} . Let $A_1^C \hookrightarrow A_2^C \hookrightarrow \dots$ be a chain in \mathcal{A} whose colimit is C and let $A_1^D \hookrightarrow A_2^D \hookrightarrow \dots$ be a chain in \mathcal{A} whose colimit is D . Moreover, let $\iota_n^C : A_n^C \hookrightarrow C$ and $\iota_n^D : A_n^D \hookrightarrow D$ be the corresponding canonical embeddings. By Lemma 1.3, for every k there is an n_k and a morphism $f_k : A_k^C \rightarrow A_{n_k}^D$ (which is necessarily an embedding) such that

$$\begin{array}{ccc} A_k^C & \xrightarrow{\iota_k^C} & C \\ f_k \downarrow & & \downarrow f \\ A_{n_k}^D & \xrightarrow{\iota_{n_k}^D} & D \end{array}$$

Without loss of generality n_k 's can be chosen in such a way that $n_1 < n_2 < \dots$. In the extension we then have

$$\begin{array}{ccc} K^0(A_k^C) & \xrightarrow{\iota_k^{K(C)}} & K(C) \\ K^0(f_k) \downarrow & & \downarrow K(f) \\ K^0(A_{n_k}^D) & \xrightarrow[\iota_{n_k}^{K(D)}]{} & K(D) \end{array}$$

whence follows that $K(f)$ is also an embedding.

Analogous argument provides a construction of the natural transformation $\eta : \text{ID} \rightarrow K$ which extends η^0 . Consider the diagram:

$$\begin{array}{ccccc}
& & \iota_n^C & & \\
& \nearrow & & \searrow & \\
A_n^C & \hookrightarrow & A_{n+1}^C & \hookrightarrow & C \\
\eta_{A_n^C}^0 \downarrow & & \eta_{A_{n+1}^C}^0 \downarrow & & \downarrow \eta_C \\
K^0(A_n^C) & \hookrightarrow & K^0(A_{n+1}^C) & \xrightarrow{\iota_{n+1}^{K(C)}} & K(C) \\
& \searrow & \nearrow & & \\
& & \iota_n^{K(C)} & &
\end{array}$$

Since C is the colimit of the chain $A_1^C \hookrightarrow A_2^C \hookrightarrow \dots$, there is a unique morphism $\eta_C : C \rightarrow K(C)$ to the tip of the competing compatible cone. The morphism η_C is clearly an embedding and it is easy to check that all the morphisms η_C constitute a natural transformation $\eta : \text{ID} \rightarrow K$. \square

We also say that K is a *Katětov functor* and from now on we denote both K and K^0 by K , and both η and η^0 by η . An obvious yet important property of K is that all its powers remain Katětov. Specifically, for $n \in \mathbb{N}$ define $\eta^n : \text{ID} \rightarrow K^n$ as $\eta_C^n = \eta_{K^{n-1}(C)} \circ \dots \circ \eta_{K(C)} \circ \eta_C : C \rightarrow K^n(C)$. Then η^n is a natural transformation witnessing that K^n is a Katětov functor. We shall elaborate this in Section 3. For now, we state the following important property of finite iterations of K .

Lemma 2.3 *Let $K : \mathcal{A} \rightarrow \mathcal{C}$ be a Katětov functor. Then for every embedding $g : A \hookrightarrow B$, where $A, B \in \text{Ob}(\mathcal{A})$, there is an $n \in \mathbb{N}$ and an embedding $h : B \hookrightarrow K^n(A)$ satisfying $h \circ g = \eta_A^n$.*

$$\begin{array}{ccc}
A & \xrightarrow{\eta_A^n} & K^n(A) \\
g \downarrow & \nearrow h & \\
B & &
\end{array}$$

Proof. If g is an isomorphism, take $n = 1$ and $h = \eta_A \circ g^{-1}$. Assume, therefore, that g is not an isomorphism. Then by Lemma 1.1(a) there exist $n \in \mathbb{N}$ and $A_1, \dots, A_n \in \text{Ob}(\mathcal{A})$ such that

$$A \hookrightarrow A_1 \hookrightarrow A_2 \hookrightarrow \dots \hookrightarrow A_n = B.$$

It is easy to see that the diagram in Fig. 1 commutes: the triangles commute by the definition of a Katětov functor, while the parallelograms commute because η is a natural transformation. So, take $h = K^{n-1}(f_1) \circ K^{n-2}(f_2) \circ \dots \circ K(f_{n-1}) \circ f_n$. \square

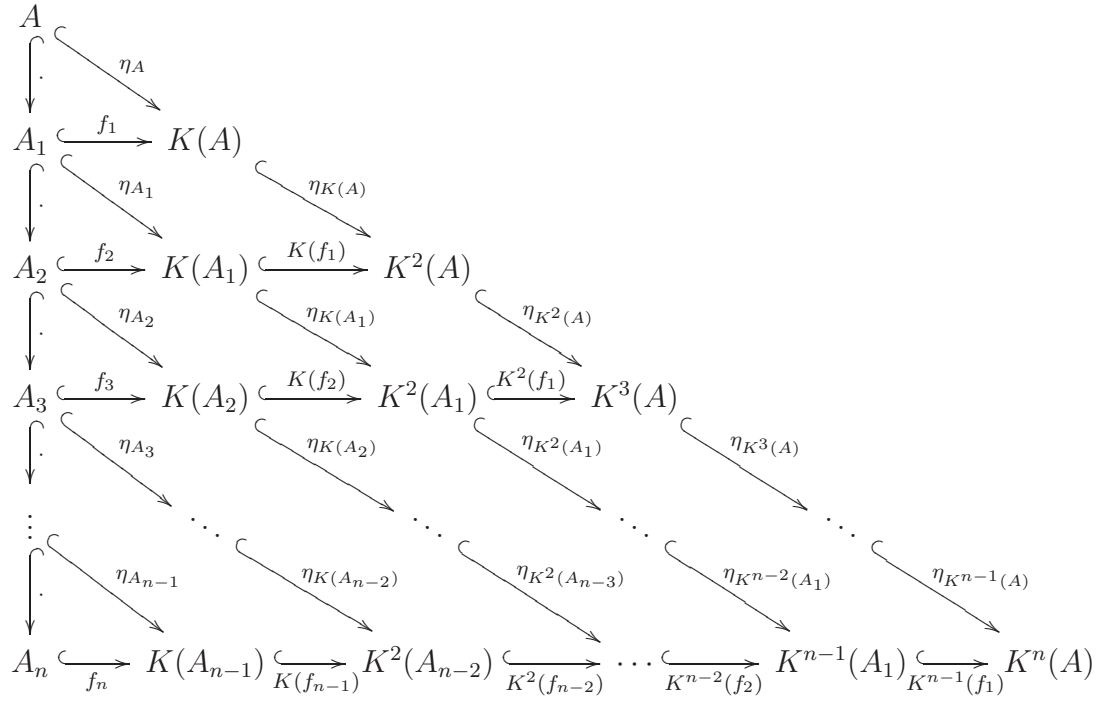


Figure 1: The proof of Lemma 2.3

2.1 Examples

Below we collect some examples of Katětov functors. The second one shows in particular that Katětov functors (as well as their powers) do not necessarily have the extension property for embeddings between objects of \mathcal{C} . In other words, Lemma 2.3 does not hold for embeddings between \mathcal{C} -objects.

Example 2.4 *A Katětov functor on the category of finite metric spaces with rational distances and nonexpansive mappings*

This is a small modification of the original Katětov functor, in order to fit into Fraïssé theory. The details are explained in Section 5 below.

Let $\mathcal{P}_2(X) = \{Y \subseteq X : |Y| = 2\}$, and let $\mathcal{P}_{\text{fin}}(X)$ denote the set of all finite subsets of X .

Example 2.5 *A Katětov functor on the category of graphs and graph homomorphisms.* Let $\langle V, E \rangle$ be a graph, where $E \subseteq \mathcal{P}_2(V)$. Put $K(\langle V, E \rangle) = \langle V^*, E^* \rangle$ where

$$\begin{aligned} V^* &= V \cup \mathcal{P}_{\text{fin}}(V), \\ E^* &= E \cup \{\{v, A\} : A \in \mathcal{P}_{\text{fin}}(V), v \in A\}. \end{aligned}$$

For a graph homomorphism $f : \langle V_1, E_1 \rangle \rightarrow \langle V_2, E_2 \rangle$ let $f^* = K(f)$ be a mapping from V_1^* to V_2^* defined by $f^*(v) = f(v)$ for $v \in V_1$ and $f^*(A) = f(A)$ for $A \in \mathcal{P}_{\text{fin}}(V_1)$. Then it is easy to show that f^* is a graph homomorphism from $\langle V_1^*, E_1^* \rangle$ to $\langle V_2^*, E_2^* \rangle$. Moreover, if f is an embedding, then so is f^* .

Now let G be an infinite graph. Let $H = G \cup \{v\}$, where v is connected to all the vertices of G . Note that each vertex of $K(G) \setminus G$ has a finite degree in $K(G)$. Thus, there is no embedding of H extending $\eta_G : G \rightarrow K(G)$. This shows that K does not have the extension property for embeddings in the bigger category consisting of all countable graphs. The same holds for K^n for every $n > 1$ (and even for its ω -power), because all “new” vertices in $K^n(G)$ have finite degrees in G .

Example 2.6 *A Katětov functor on the category of K_n -free graphs and graph embeddings.* Fix an integer $n \geq 3$. Let $\langle V, E \rangle$ be a K_n -free graph, where E is the set of some 2-element subsets of V . Put $K(\langle V, E \rangle) = \langle V^*, E^* \rangle$ where

$$\begin{aligned} V^* &= V \cup V', \\ V' &= \{A \in \mathcal{P}_{\text{fin}}(V) : \langle A, E \cap \mathcal{P}_2(A) \rangle \text{ is } K_{n-1}\text{-free}\}, \\ E^* &= E \cup \{\{v, A\} : A \in V', v \in A\}. \end{aligned}$$

For a graph embedding $f : \langle V_1, E_1 \rangle \hookrightarrow \langle V_2, E_2 \rangle$ let $f^* = K(f)$ be a mapping from V_1^* to V_2^* defined by $f^*(v) = f(v)$ for $v \in V_1$ and $f^*(A) = f(A)$ for $A \in V'_1$. Then it is easy to show that f^* is a graph embedding from $\langle V_1^*, E_1^* \rangle$ to $\langle V_2^*, E_2^* \rangle$.

Example 2.7 *A Katětov functor on the category of digraphs and digraph homomorphisms.* Let $\langle V, E \rangle$ be a digraph, where $E \subseteq V^2$ is an irreflexive relation satisfying $(x, y) \in E \Rightarrow (y, x) \notin E$. Put $K(\langle V, E \rangle) = \langle V^*, E^* \rangle$ where

$$\begin{aligned} V^* &= V \cup V', \\ V' &= \{ \langle A, B \rangle : A, B \in \mathcal{P}_{\text{fin}}(V) \text{ such that } A \cap B = \emptyset \}, \\ E^* &= E \cup \{ \langle v, \langle A, B \rangle \rangle : v \in V, \langle A, B \rangle \in V', v \in A \} \\ &\quad \cup \{ \langle \langle A, B \rangle, v \rangle : v \in V, \langle A, B \rangle \in V', v \in B \}. \end{aligned}$$

For a digraph homomorphism $f : \langle V_1, E_1 \rangle \rightarrow \langle V_2, E_2 \rangle$ let $f^* = K(f)$ be a mapping from V_1^* to V_2^* defined by $f^*(v) = f(v)$ for $v \in V_1$ and $f^*(\langle A, B \rangle) = \langle f(A), f(B) \rangle$ for $\langle A, B \rangle \in V_1'$. Then it is easy to show that f^* is a digraph homomorphism from $\langle V_1^*, E_1^* \rangle$ to $\langle V_2^*, E_2^* \rangle$. Moreover, if f is an embedding, then so is f^* .

Example 2.8 *A Katětov functor on the category of all finite linear orders and monotone mappings.* For a linear order $\langle A, \leq \rangle$ put $K(\langle A, \leq \rangle) = \langle A^*, \leq^* \rangle$ where

$$\begin{aligned} A^* &= A \cup A', \\ A' &= \{ \langle U, V \rangle : \{U, V\} \text{ is a partition of } A \text{ and } \forall u \in U \forall v \in V (u \leq v) \}, \\ \leq^* &= \leq \cup \{ \langle a, \langle U, V \rangle \rangle : a \in U \} \cup \{ \langle \langle U, V \rangle, a \rangle : a \in V \} \\ &\quad \cup \{ \langle \langle U_1, V_1 \rangle, \langle U_2, V_2 \rangle \rangle : V_1 \cap U_2 \neq \emptyset \} \\ &\quad \cup \{ \langle \langle U, V \rangle, \langle U, V \rangle \rangle : \langle U, V \rangle \in A' \}. \end{aligned}$$

Then it is easy to see that \leq^* is a linear order on A^* . For a monotone map $f : \langle A_1, \leq_1 \rangle \rightarrow \langle A_2, \leq_2 \rangle$ let $f^* = K(f)$ be the mapping from A_1^* to A_2^* defined by $f^*(a) = f(a)$ for $a \in A_1$ and for $\langle U, V \rangle \in A_1'$ we put $f^*(\langle U, V \rangle) = \langle A_2 \setminus W, W \rangle$ where $W = f(V)$. Then it is easy to show that f^* is a monotone map from $\langle A_1^*, \leq_1^* \rangle$ to $\langle A_2^*, \leq_2^* \rangle$. Moreover, if f is an embedding, then so is f^* .

The description of $K(\langle A, \leq \rangle)$ in case $\langle A, \leq \rangle$ is a countable linear order is more involved. As an illustration, let us just say that $K(\langle \mathbb{Q}, \leq \rangle)$ is of the form $Q_1 \cup Q_2$, where both Q_1 and Q_2 are dense in $K(\langle \mathbb{Q}, \leq \rangle)$ and Q_1 serves as a copy of $\langle \mathbb{Q}, \leq \rangle$ while Q_2 serves as the set of all one-point extensions of finite subsets of Q_1 , ordered in a suitable way.

Example 2.9 *A Katětov functor on the category of partially ordered sets and monotone mappings.* For a partially ordered set $\langle A, \leq \rangle$ put $K(\langle A, \leq \rangle) = \langle A^*, \leq^* \rangle$ where

$$\begin{aligned} A^* &= A \cup A', \\ A' &= \{ \langle U, V \rangle : U, V \in \mathcal{P}_{\text{fin}}(A) \text{ and } \forall u \in U \forall v \in V (u \leq v) \}, \\ \leq^* &= \leq \cup \{ \langle a, \langle U, V \rangle \rangle : \exists u \in U (a \leq u) \} \\ &\quad \cup \{ \langle \langle U, V \rangle, a \rangle : \exists v \in V (v \leq a) \} \\ &\quad \cup \{ \langle \langle U_1, V_1 \rangle, \langle U_2, V_2 \rangle \rangle : \exists v \in V_1 \exists u \in U_2 (v \leq u) \} \\ &\quad \cup \{ \langle \langle U, V \rangle, \langle U, V \rangle \rangle : \langle U, V \rangle \in A' \}. \end{aligned}$$

Then it is easy to see that \leq^* is a partial order on A^* . For a monotone map $f : \langle A_1, \leq_1 \rangle \rightarrow \langle A_2, \leq_2 \rangle$ let $f^* = K(f)$ be a mapping from A_1^* to A_2^* defined by $f^*(a) = f(a)$ for $a \in A_1$ and for $\langle U, V \rangle \in A_1'$ we put $f^*(\langle U, V \rangle) = \langle f(U), f(V) \rangle$. Then it is easy to show that f^* is a monotone map from $\langle A_1^*, \leq_1^* \rangle$ to $\langle A_2^*, \leq_2^* \rangle$. Moreover, if f is an embedding, then so is f^* .

Example 2.10 *A Katětov functor on the category of tournaments and embeddings.* Recall that a tournament is a digraph $\langle V, E \rangle$ such that for every $x, y \in V$ exactly one of the possibilities holds: either $x = y$ or $(x, y) \in E$ or $(y, x) \in E$.

For a finite set A and a positive integer n let $A^{\leq n}$ be the set of all sequences $\langle a_1, \dots, a_k \rangle$ of elements of A where $k \in \{0, 1, \dots, n\}$. In case of $k = 0$ we actually have the empty sequence $\langle \rangle$, as we will be careful to distinguish the 1-element sequence $\langle a \rangle$ from $a \in A$. For a sequence $s \in A^{\leq n}$ let $|s|$ denote the length of s . For a tournament $T = \langle V, E \rangle$, where $E \subseteq V^2$, let $n = |V|$ and let $T^{\leq n}$ be the tournament whose set of vertices is $V^{\leq n}$ and whose set of edges is defined *lexicographically* as follows:

- if s and t are sequences such that $|s| < |t|$, put $s \rightarrow t$ in $T^{\leq n}$;
- if $s = \langle s_1, \dots, s_k \rangle$ and $t = \langle t_1, \dots, t_k \rangle$ are distinct sequences of the same length, find the smallest i such that $s_i \neq t_i$ and then put $s \rightarrow t$ in $T^{\leq n}$ if and only if $s_i \rightarrow t_i$ in T .

For a tournament $T = \langle V, E \rangle$ put $K(T) = \langle V^*, E^* \rangle$ where

$$\begin{aligned} V^* &= V \cup V^{\leq n}, \\ E^* &= E \cup E(T^{\leq n}) \cup \{ \langle v, s \rangle : v \in V, s \in V^{\leq n}, v \text{ appears as an entry in } s \} \\ &\quad \cup \{ \langle s, v \rangle : v \in V, s \in V^{\leq n}, v \text{ does not appear as an entry in } s \}. \end{aligned}$$

Then it is easy to see that $\langle V^*, E^* \rangle$ is a tournament realizing all one-point extensions of $\langle V, E \rangle$. For an embedding $f : \langle V_1, E_1 \rangle \rightarrow \langle V_2, E_2 \rangle$ let $f^* = K(f)$ be the mapping from V_1^* to V_2^* defined by $f^*(v) = f(v)$ for $v \in V_1$ and for $\langle s_1, \dots, s_k \rangle \in V_1^{\leq n}$ we put $f^*(\langle s_1, \dots, s_k \rangle) = \langle f(s_1), \dots, f(s_k) \rangle$. Then it is easy to show that f^* is an embedding from $\langle V_1^*, E_1^* \rangle$ to $\langle V_2^*, E_2^* \rangle$. Finally, K is a Katětov functor which is witnessed by the obvious natural transformation mapping $T = \langle V, E \rangle$ to its copy in $K(T)$.

Example 2.11 *A Katětov functor on the category of all Boolean algebras.* For a finite set A let $B(A)$ denote the finite Boolean algebra whose set of atoms is A . For a finite Boolean algebra $B(A)$ put $K(B(A)) = B(\{0, 1\} \times A)$ and let $\eta_{B(A)} : B(A) \hookrightarrow B(\{0, 1\} \times A)$ be the unique homomorphism which takes $a \in A$ to $\langle 0, a \rangle \vee \langle 1, a \rangle \in B(\{0, 1\} \times A)$. Clearly, $\eta_{B(A)}$ is an embedding. Let us define K on homomorphisms between finite Boolean algebras as follows. Let $f : B(A) \rightarrow B(A')$ be a homomorphism and assume that for $a \in A$ we have $f(a) = \bigvee S(a)$ for some $S(a) \subseteq A'$, with the convention that $\bigvee \emptyset = 0$. Then for $i \in \{0, 1\}$ let

$K(f)(\langle i, a \rangle) = \bigvee(\{i\} \times S(a))$. This turns K into a functor from the category of finite Boolean algebras into itself which preserves embeddings and such that $\eta : \text{ID} \rightarrow K$ is a natural transformation.

Let us show that K is indeed a Katětov functor. Let $j : B(A) \hookrightarrow B(A')$. Then $B(A') = \langle B(A) \cup \{x\} \rangle$ since $B(A')$ is a one-point extension of $B(A)$, so $A' = (\bigcup_{a \in A} \{x \wedge j(a), \bar{x} \wedge j(a)\}) \setminus \{0\}$. Let $g : B(A') \hookrightarrow K(B(A))$ be the embedding defined on the atoms of $B(A')$ as follows:

- if $x \wedge j(a) = j(a)$ (and consequently $\bar{x} \wedge j(a) = 0$) or $x \wedge j(a) = 0$ (and consequently $\bar{x} \wedge j(a) = j(a)$) let g take $j(a)$ to $\langle 0, a \rangle \vee \langle 1, a \rangle$,
- if $x \wedge a \neq 0$ and $\bar{x} \wedge a \neq 0$ let g take $x \wedge a$ to $\langle 0, a \rangle$ and $\bar{x} \wedge a$ to $\langle 1, a \rangle$,

and which extends to the rest of $B(A')$ in the obvious way. Then it is easy to see that $g \circ j = \eta_{B(A)}$.

2.2 Sufficient conditions for the existence of Katětov functors

Let Δ be a purely relational language, let A be a Δ -structure, and let B_1, B_2 be Δ -structures such that A is a substructure of both of them and $A = B_1 \cap B_2$. The *free amalgam* of the B_1, B_2 over A is the Δ -structure C with universe $B_1 \cup B_2$ such that both B_1, B_2 are substructures of C and for every $R \in \Delta$ we have that $R^C = R^{B_1} \cup R^{B_2}$ (in other words, no tuple which meets $B_1 \setminus A$ and $B_2 \setminus A$ satisfies any relation symbol in Δ). Following [3], we say that \mathcal{A} has the *free amalgamation property* if every triple A, B_1, B_2 as above has the free amalgam in \mathcal{A} . The next result is implicit in [3] (see Definition 3.7 in [3] and the comment that follows).

Theorem 2.12 (implicit in [3]) *If \mathcal{A} has free amalgamations then a Katětov functor $K : \mathcal{A} \rightarrow \mathcal{C}$ exists.*

The following theorem is a strengthening of this as well as of the main result of [8]. We say that \mathcal{A} has *one-point extension pushouts* [resp. *mixed pushouts*] in \mathcal{C} if for every morphism $f : A_0 \rightarrow A_1$ in \mathcal{A} and a one-point extension [resp. embedding] $g : A_0 \hookrightarrow A_2$ in \mathcal{A} there exists a $B \in \text{Ob}(\mathcal{A})$, an embedding $p : A_1 \hookrightarrow B$ and a morphism $q : A_2 \rightarrow B$ such that $p \circ f = q \circ g$ and this commuting square is a pushout square in the category \mathcal{C}^{hom} of *all* homomorphisms between \mathcal{C} -objects.

$$\begin{array}{ccc} A_0 & \xrightarrow[g]{} & A_2 \\ f \downarrow & & \downarrow q \\ A_1 & \xrightarrow[p]{} & B \end{array}$$

Note that free amalgamations are particular examples of pushouts. Note also that typical categories of models with embeddings rarely have pushouts. Namely, recall

that a pair of morphisms (p, q) provides the pushout of (f, g) if $p \circ f = q \circ g$ and for every other pair (p', q') satisfying $p' \circ f = q' \circ g$ there exists a unique morphism h such that $h \circ p = p'$ and $h \circ q = q'$. Now, if $f = g$ and (p', q') consists of identities then clearly h cannot be an embedding. That is why, in the definition above, we have to consider pushouts in the category of all homomorphisms.

Lemma 2.13 *Suppose*

$$\begin{array}{ccc} A_0 & \xhookrightarrow{g} & A_2 \\ f \downarrow & & \downarrow q \\ A_1 & \xhookrightarrow{p} & B \end{array}$$

is a pushout square in the category \mathcal{C}^{hom} of all homomorphisms. If g is a one-point extension then so is p .

Proof. Let B_1 be the substructure of B generated by $p[A_1] \cup \{q(s)\}$, where $s \in A_2$ is such that $g[A_0] \cup \{s\}$ generates A_2 . Notice that $q[A_2] \subseteq B_1$. In other words, the square

$$\begin{array}{ccc} A_0 & \xhookrightarrow{g} & A_2 \\ f \downarrow & & \downarrow q_1 \\ A_1 & \xhookrightarrow{p_1} & B_1 \end{array}$$

is commutative, where p_1 and q_1 denote the same mappings as p and q , respectively. By the universality of a pushout, there is a unique homomorphism $h: B \rightarrow B_1$ such that $h \circ p = p_1$ and $h \circ q = q_1$. Let h_1 the composition of h with the inclusion $B_1 \subseteq B$. Again by the universality of a pushout, $h_1: B \rightarrow B$ is the unique homomorphism satisfying $h_1 \circ p = p$ and $h_1 \circ q = q$. It follows that $h_1 = \text{id}_B$ and hence $B_1 = B$. This completes the proof. \square

It turns out that both variants of the definition above are equivalent. In practice however, it is usually easier to verify the existence of pushouts for one-point extensions.

Proposition 2.14 *The following properties are equivalent:*

- (a) \mathcal{A} has the one-point extension pushouts in \mathcal{C} .
- (b) \mathcal{A} has mixed pushouts in \mathcal{C} .

Proof. Only implication (a) \implies (b) requires a proof. Fix $f: A_0 \rightarrow A_1$ and $g: A_0 \hookrightarrow A_2$ as above and assume that $g = g_n \circ \dots \circ g_1$ is the composition of n one-point extensions $g_i: E_i \hookrightarrow E_{i+1}$, where $E_1 = A_0$, $E_{n+1} = A_2$. By Lemma 2.13

we have the following sequence of pushout squares in \mathcal{C}^{hom} .

$$\begin{array}{ccccccc}
A_0 & \xrightarrow{g_1} & E_2 & \xrightarrow{g_2} & E_3 & \xrightarrow{\quad} & \dots \xrightarrow{\quad} E_n \xrightarrow{g_n} A_2 \\
f \downarrow & & \downarrow q_1 & & \downarrow q_2 & & \downarrow q_{n-1} & \downarrow q_n \\
A_1 & \xrightarrow{p_1} & B_2 & \xrightarrow{p_2} & B_3 & \xrightarrow{\quad} & \dots \xrightarrow{\quad} B_n \xrightarrow{p_n} B
\end{array}$$

Clearly, the composition of all these squares is a pushout in \mathcal{C}^{hom} . \square

Theorem 2.15 *If \mathcal{A} has one-point extension pushouts in \mathcal{C} then a Katětov functor $K : \mathcal{A} \rightarrow \mathcal{C}$ exists.*

Proof. Let us first show that every countable source $(A \hookrightarrow B_n)_{n \in \mathbb{N}}$ has a pushout in \mathcal{C} , where $A, B_1, B_2, \dots \in \text{Ob}(\mathcal{A})$. Let $e_n : A \hookrightarrow B_n$ be the embeddings in this source. Let $P_2 \in \text{Ob}(\mathcal{A})$ together with the embeddings $f_2 : B_1 \hookrightarrow P_2$ and $g_2 : B_2 \hookrightarrow P_2$ be the pushout of e_1 and e_2 . Next, let $P_3 \in \text{Ob}(\mathcal{A})$ together with the embeddings $f_3 : P_2 \hookrightarrow P_3$ and $g_3 : B_3 \hookrightarrow P_3$ be the pushout of $f_2 \circ e_1$ and e_3 . Then, let $P_4 \in \text{Ob}(\mathcal{A})$ together with the embeddings $f_4 : P_3 \hookrightarrow P_4$ and $g_4 : B_4 \hookrightarrow P_4$ be the pushout of $f_3 \circ f_2 \circ e_1$ and e_4 , and so on:

$$\begin{array}{ccccccc}
B_1 & \xrightarrow{f_2} & P_2 & \xrightarrow{f_3} & P_3 & \xrightarrow{f_4} & P_4 \xrightarrow{f_5} \dots \\
\uparrow e_1 & & \uparrow g_2 & & \uparrow g_3 & & \uparrow g_4 \\
A & \xrightarrow{e_2} & B_2 & & B_3 & & B_4 \\
& \searrow e_3 & & \searrow e_4 & & & \\
& & & & & &
\end{array}$$

Let $P \in \text{Ob}(\mathcal{C})$ be the colimit of the chain $B_1 \hookrightarrow P_2 \hookrightarrow P_3 \hookrightarrow P_4 \hookrightarrow \dots$. It is easy to show that P is the pushout of the source $(A \hookrightarrow B_n)_{n \in \mathbb{N}}$.

Let us now construct the Katětov functor as the pushout of all the one-point extensions of an object in \mathcal{A} . More precisely, for every $A \in \text{Ob}(\mathcal{A})$ let us fix embeddings $e_n : A \hookrightarrow B_n$, where B_1, B_2, \dots is the list of all the one-point extensions of A , where every isomorphism type is taken exactly once to keep the list countable. Define $K(A)$ to be the pushout of the source $(e_n : A \hookrightarrow B_n)_n$. This is how K acts on objects.

Let us show how K acts on morphisms. Take any morphism $h : A \rightarrow A'$ in \mathcal{A} . Let $(e_i : A \hookrightarrow B_i)_{i \in I}$ be the source consisting of all the one-point extensions of A (with every isomorphism type is taken exactly once), and let $(e'_j : A' \hookrightarrow B'_j)_{j \in J}$ be the source consisting of all the one-point extensions of A' (with every isomorphism type is taken exactly once). By the assumption, for every $i \in I$ there exists an $m(i) \in J$ and a morphism $h_i : B_i \rightarrow B'_{m(i)}$ such that the following is a pushout

square in \mathcal{C} :

$$\begin{array}{ccc} A & \xrightarrow[e_i]{} & B_i \\ h \downarrow & & \downarrow h_i \\ A' & \xrightarrow[e'_{m(i)}]{} & B'_{m(i)} \end{array}$$

Now, $K(A')$ is a pushout of the source $(e'_j : A' \hookrightarrow B'_j)_{j \in J}$ so let us denote the canonical embeddings $B'_j \hookrightarrow K(A')$ by ι'_j , $j \in J$. Therefore, $(\iota'_{m(i)} \circ h_i : B_i \rightarrow K(A'))_{i \in I}$ is a compatible cone over the source $(e_i : A \hookrightarrow B_i)_{i \in I}$, so there is a unique mediating morphism $\tilde{h} : K(A) \rightarrow K(A')$. Then we put $K(h) = \tilde{h}$. \square

Note that the category of graphs and homomorphisms has pushouts, while the category of K_n -free graphs has pushouts of embeddings only. On the other hand, categories like tournaments or linear orderings do not have pushouts, even when considering all homomorphisms.

3 Katětov construction

Definition 3.1 Let $K : \mathcal{C} \rightarrow \mathcal{C}$ be a Katětov functor. A *Katětov construction* is a chain of the form:

$$C \xrightarrow{\eta_C} K(C) \xrightarrow{\eta_{K(C)}} K^2(C) \xrightarrow{\eta_{K^2(C)}} K^3(C) \hookrightarrow \dots$$

where $C \in \text{Ob}(\mathcal{C})$. We denote the colimit of this chain by $K^\omega(C)$. An object $L \in \text{Ob}(\mathcal{C})$ can be obtained by the Katětov construction starting from C if $L = K^\omega(C)$. We say that L can be obtained by the Katětov construction if $L = K^\omega(C)$ for some $C \in \text{Ob}(\mathcal{C})$.

Note that K^ω is actually a functor from \mathcal{C} into \mathcal{C} . Namely, for a morphism $f : A \rightarrow B$ let $K^\omega(f)$ be the unique morphism $K^\omega(A) \rightarrow K^\omega(B)$ from the colimit of the Katětov construction starting from A to the competitive compatible cone with the tip at $K^\omega(B)$ and morphisms $(\hookrightarrow \circ K^n(f))_{n \in \mathbb{N}}$:

$$\begin{array}{ccccccc} & & & & & & K^\omega(A) \\ & & & & & \nearrow & \\ & & & & \eta_A^\omega & \nearrow & \\ A & \xrightarrow{\eta_A} & K(A) & \xrightarrow{\eta_{K(A)}} & K^2(A) & \xrightarrow{\eta_{K^2(A)}} & \dots \\ & \searrow & \downarrow K(f) & \searrow & \downarrow K^2(f) & \searrow & \\ B & \xrightarrow{\eta_B} & K(B) & \xrightarrow{\eta_{K(B)}} & K^2(B) & \xrightarrow{\eta_{K^2(B)}} & \dots \\ & \searrow & \downarrow K(f) & \searrow & \downarrow K^2(f) & \searrow & \\ & & & & & & K^\omega(B) \end{array}$$

η_B^ω (curved arrow from B to $K^\omega(B)$)
 η_A^ω (curved arrow from A to $K^\omega(A)$)
 $K^\omega(f)$ (dotted arrow from $K^\omega(A)$ to $K^\omega(B)$)

It is clear that K^ω preserves embeddings (the colimit of embeddings is an embedding). Moreover, the canonical embeddings $\eta_A^\omega : A \hookrightarrow K^\omega(A)$ constitute a natural transformation $\eta^\omega : \text{ID} \rightarrow K^\omega$. Thus, we have:

Theorem 3.2 $K^\omega : \mathcal{C} \rightarrow \mathcal{C}$ is a Katětov functor.

Recall that a countable structure L is *ultrahomogeneous* if every isomorphism between two finitely generated substructures of L extends to an automorphism of L . More precisely, L is ultrahomogeneous if for all $A, B \in \text{age}(L)$, embeddings $j_A : A \hookrightarrow L$ and $j_B : B \hookrightarrow L$, and for every isomorphism $f : A \rightarrow B$ there is an automorphism f^* of L such that $j_B \circ f = f^* \circ j_A$.

$$\begin{array}{ccc} A & \xhookrightarrow{j_A} & L \\ f \downarrow & & \downarrow f^* \\ B & \xhookrightarrow{j_B} & L \end{array}$$

One of the crucial points of the classical Fraïssé theory is the fact that every ultrahomogeneous structure is the Fraïssé limit of its age, and every Fraïssé limit is ultrahomogeneous.

Analogously, we say that a countable structure L is *\mathcal{C} -morphism-homogeneous*, if every \mathcal{C} -morphism between two finitely generated substructures of L extends to a \mathcal{C} -endomorphism of L . More precisely, L is \mathcal{C} -morphism-homogeneous if for all $A, B \in \text{age}(L)$, embeddings $j_A : A \hookrightarrow L$ and $j_B : B \hookrightarrow L$, and for every \mathcal{C} -morphism $f : A \rightarrow B$ there is a \mathcal{C} -endomorphism f^* of L such that $j_B \circ f = f^* \circ j_A$. In particular, if \mathcal{C} is the category of all countable Δ -structures with all homomorphisms between them, instead of saying that L is \mathcal{C} -morphism-homogeneous, we say that L is *homomorphism-homogeneous*. The study of homomorphism-homogeneity was initiated by Cameron & Nešetřil [5].

The first part of the next result can be viewed as an analogy to Banach's contraction principle: iterating a Katětov functor, starting from an arbitrary object, one always “tends” to the Fraïssé limit, which can be regarded as a “fixed point” of the Katětov functor.

Theorem 3.3 *If there exists a Katětov functor $K : \mathcal{A} \rightarrow \mathcal{C}$, then \mathcal{A} is an amalgamation class, it has a Fraïssé limit L in \mathcal{C} , and L can be obtained by the Katětov construction starting from an arbitrary $C \in \text{Ob}(\mathcal{C})$. Moreover, L is \mathcal{C} -morphism-homogeneous.*

Proof. Take any $C \in \text{Ob}(\mathcal{C})$, let

$$C \xhookrightarrow{\eta_C} K(C) \xhookrightarrow{\eta_{K(C)}} K^2(C) \xhookrightarrow{\eta_{K^2(C)}} K^3(C) \hookrightarrow \dots \quad (2)$$

be the Katětov construction starting from C , and let $L \in \text{Ob}(\mathcal{C})$ be the colimit of this chain. Let $\iota_n : K^n(C) \hookrightarrow L$ be the canonical embeddings of the colimit diagram.

Let us first show that $\text{age}(L) = \text{Ob}(\mathcal{A})$. Lemma 1.4 yields $\text{age}(L) \subseteq \text{Ob}(\mathcal{A})$, so let us show that $\text{Ob}(\mathcal{A}) \subseteq \text{age}(L)$. Take any $B \in \text{Ob}(\mathcal{A})$ and let $A_1 \hookrightarrow A_2 \hookrightarrow \dots$ be a chain whose colimit is C . Since \mathcal{A} has (JEP) there is a $D \in \text{Ob}(\mathcal{A})$ such that $A_1 \hookrightarrow D \hookleftarrow B$. Lemma 2.3 then ensures that there is an $n \in \mathbb{N}$ such that $D \hookrightarrow K^n(A_1)$. On the other hand, $A_1 \hookrightarrow C$ implies $K^n(A_1) \hookrightarrow K^n(C)$. Therefore, $B \hookrightarrow D \hookrightarrow K^n(A_1) \hookrightarrow K^n(C) \hookrightarrow L$, so $B \in \text{age}(L)$. This completes the proof that $\text{age}(L) = \text{Ob}(\mathcal{A})$.

Next, let us show that L realizes all one-point extensions, that is, let us show that for all $A, B \in \text{Ob}(\mathcal{A})$ such that $A \hookrightarrow B$ and every embedding $f : A \hookrightarrow L$ there is an embedding $g : B \hookrightarrow L$ such that:

$$\begin{array}{ccc} A & \xhookrightarrow{f} & L \\ \downarrow \cdot & \nearrow g & \\ B & & \end{array} \quad (3)$$

Take any $A, B \in \text{Ob}(\mathcal{A})$ such that $A \hookrightarrow B$ and let $f : A \hookrightarrow L$ be an arbitrary embedding. By Lemma 1.3 there is an $n \in \mathbb{N}$ and an embedding $h : A \hookrightarrow K^n(C)$ such that $f \circ h = \iota_n$. Note that the following diagram commutes:

$$\begin{array}{ccccc} A & \xhookrightarrow{h} & K^n(C) & \xhookrightarrow{\iota_n} & L \\ \downarrow \cdot & \searrow \eta_A & \searrow \eta_{K^n(C)} & & \uparrow \iota_{n+1} \\ B & \xhookrightarrow{j} & K(A) & \xhookrightarrow{K(h)} & K^{n+1}(C) \end{array}$$

(the triangle on the left commutes due to the definition of the Katětov functor, the parallelogram in the middle commutes because η is a natural transformation, while the triangle on the right commutes as part of the colimit diagram for the chain (2)). Let $g = \iota_{n+1} \circ K(h) \circ j$. Having in mind that $f = \iota_n \circ h$, from the last commuting diagram we immediately get that the diagram (3) commutes for this particular choice of g .

Therefore, L realizes all one-point extensions, so L is an ultrahomogeneous countable structure whose age is $\text{Ob}(\mathcal{A})$. Consequently, L is the Fraïssé limit of $\text{Ob}(\mathcal{A})$, whence we easily conclude that \mathcal{A} is an amalgamation class. Moreover, the Fraïssé limit of \mathcal{A} can be obtained by the Katětov construction starting from an arbitrary $C \in \text{Ob}(\mathcal{C})$.

Finally, let us show that L is \mathcal{C} -morphism-homogeneous. Take any $A, B \in \text{age}(L)$, fix embeddings $j_A : A \hookrightarrow L$ and $j_B : B \hookrightarrow L$, and let $f : A \rightarrow B$ be an arbitrary morphism. Then

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \eta_A^\omega \downarrow & & \downarrow \eta_B^\omega \\ K^\omega(A) & \xrightarrow{K^\omega(f)} & K^\omega(B) \end{array} \quad (4)$$

Having in mind that $K^\omega(A)$ and $K^\omega(B)$ are colimits of Katětov constructions starting from A and B , respectively, we conclude that both $K^\omega(A)$ and $K^\omega(B)$ are isomorphic to L . Since L is ultrahomogeneous, there exist isomorphisms $s : K^\omega(A) \rightarrow L$ and $t : K^\omega(B) \rightarrow L$ such that

$$\begin{array}{ccc} & A & \\ \eta_A^\omega \swarrow & \downarrow j_A & \\ K^\omega(A) & \xrightarrow{s} & L \end{array} \quad \begin{array}{ccc} & B & \\ j_B \downarrow & \nwarrow \eta_B^\omega & \\ L & \xleftarrow{t} & K^\omega(B) \end{array} \quad (5)$$

Putting diagrams (4) and (5) together we obtain

$$\begin{array}{ccccc} & A & \xrightarrow{f} & B & \\ \eta_A^\omega \swarrow & \downarrow j_A & & \downarrow j_B & \nwarrow \eta_B^\omega \\ K^\omega(A) & \xrightarrow{s} & L & \xrightarrow{\quad f^* \quad} & L & \xleftarrow{t} & K^\omega(B) \\ & & \searrow K^\omega(f) & & \end{array}$$

whence follows that $f^* = t \circ K^\omega(f) \circ s^{-1}$ is a \mathcal{C} -endomorphism of L which extends f . So, L is \mathcal{C} -morphism-homogeneous. \square

Consequently, if the Katětov functor is defined on a category of countable Δ -structures and all homomorphisms between Δ -structures, the Fraïssé limit of \mathcal{A} is both ultrahomogeneous and homomorphism-homogeneous.

Example 3.4 Let $n \geq 3$ be an integer, let \mathcal{C}_n be the category of all countable K_n -free graphs together with all graph homomorphisms, and let \mathcal{A}_n be the full subcategory of \mathcal{C}_n spanned by all finite K_n -free graphs. Then there does not exist a Katětov functor $K : \mathcal{A}_n \rightarrow \mathcal{C}_n$, for if there were one, the Henson graph H_n – the Fraïssé limit of \mathcal{A}_n – would be homomorphism-homogeneous, and we know this is not the case.

(Proof. Since H_n is universal for all finite K_n -free graphs, it embeds both K_{n-1} and the star S_n , which is the graph where one vertex is adjacent to $n-1$ independent vertices. Let f be a partial homomorphism of H_n which maps the $n-1$ independent vertices of the star S_n onto the vertices of K_{n-1} . If H_n were homomorphism-homogeneous, f would extend to an endomorphism f^* of H_n , so f^* applied to the center of the star S_n would produce a vertex adjacent to each of the vertices of K_{n-1} inducing thus a K_n in H_n , which is not possible.)

Note however that there exists a Katětov functor from the category \mathcal{A}'_n of all finite K_n -free graphs together with all graph embeddings to the category \mathcal{C}'_n of all countable K_n -free graphs together with all graph embeddings (see Example 2.6).

3.1 Characterizations of the existence of a Katětov functor

The following theorem gives a necessary and sufficient condition for a Katětov functor to exist. It depends on a condition that resembles the Herwig-Lascar-Solecki property (see [11, 18]).

Definition 3.5 A *partial morphism* of $C \in \text{Ob}(\mathcal{C})$ is a triple $\langle A, f, B \rangle$ where $A, B \leq C$ are finitely generated and $f : A \rightarrow B$ is a \mathcal{C} -morphism. We say that $C \in \text{Ob}(\mathcal{C})$ has the *morphism extension property in \mathcal{C}* if for any choice f_1, f_2, \dots of partial morphisms of C there exist $D \in \text{Ob}(\mathcal{C})$ and $m_1, m_2, \dots \in \text{End}(D)$ such that C is a substructure of D , m_i is an extension of f_i for all i , and the following *coherence* conditions are satisfied for all i, j and k :

- if $f_i = \langle A, \text{id}_A, A \rangle$ then $m_i = \text{id}_D$,
- if f_i is an embedding, then so is m_i , and
- if $f_i \circ f_j = f_k$ then $m_i \circ m_j = m_k$.

We say that \mathcal{C} has the *morphism extension property* if every $C \in \text{Ob}(\mathcal{C})$ has the morphism extension property in \mathcal{C} .

Theorem 3.6 *The following are equivalent:*

- (1) *there exists a Katětov functor $K : \mathcal{A} \rightarrow \mathcal{C}$;*
- (2) *\mathcal{A} has (AP) and \mathcal{C} has the morphism extension property;*
- (3) *\mathcal{A} has (AP) and the Fraïssé limit of \mathcal{A} has the morphism extension property in \mathcal{C} .*

Proof. (1) \Rightarrow (2): From Theorem 3.3 we know that \mathcal{A} is an amalgamation class, it has a Fraïssé limit L in \mathcal{C} , and L can be obtained by the Katětov construction starting from an arbitrary $C \in \text{Ob}(\mathcal{C})$. Now, take any $C \in \text{Ob}(\mathcal{C})$ and let us show that C has the morphism extension property in \mathcal{C} . Since L is universal for $\text{Ob}(\mathcal{C})$, without loss of generality we can assume that $C \leq L$. For every finitely generated $A \leq C$ fix an isomorphism $j_A : K^\omega(A) \rightarrow L$ such that

$$\begin{array}{ccc} A & \xhookrightarrow{\eta_A^\omega} & K^\omega(A) \\ \downarrow \leq & & \downarrow j_A \\ C & \xhookrightarrow{\leq} & L \end{array}$$

(such an isomorphism exists because L is ultrahomogeneous). Now, for any family $\langle A_i, f_i, B_i \rangle$, $i \in I$, of partial morphisms of C it is easy to see that L together with its

endomorphisms $m_i = j_{B_i} \circ K^\omega(f_i) \circ j_{A_i}^{-1}$, $i \in I$, is an extension of C and its partial morphisms f_i , $i \in I$:

$$\begin{array}{ccccc} A_i & \xrightarrow{\eta_{A_i}^\omega} & K^\omega(A_i) & \xrightarrow{j_{A_i}} & L \\ f_i \downarrow & & K^\omega(f_i) \downarrow & & \downarrow m_i \\ B_i & \xrightarrow{\eta_{B_i}^\omega} & K^\omega(B_i) & \xrightarrow{j_{B_i}} & L \end{array}$$

The coherence requirements are satisfied since K^ω is a functor which preserves embeddings.

(2) \Rightarrow (3): Trivial.

(3) \Rightarrow (1): Let L be the Fraïssé limit of \mathcal{A} . For every $A \in \text{Ob}(\mathcal{A})$ fix an embedding $j_A : A \hookrightarrow L$. Then every \mathcal{A} -morphism $f : A \rightarrow B$ induces a partial morphism $p(f) : j_A(A) \rightarrow j_B(B)$ of L by $p(f) = j_B \circ f \circ j_A^{-1}$. Since L is a countable structure, there are only countably many partial morphisms $p(f)$, say, p_1, p_2, \dots . By the assumption of (3) there exist $D \in \text{Ob}(\mathcal{C})$ and $m_1, m_2, \dots \in \text{End}(D)$ such that L is a substructure of D , m_i is an extension of p_i for all i , and the coherence conditions are satisfied. Let $e : L \hookrightarrow D$ be the inclusion of L into D .

Define a functor $K : \mathcal{A} \rightarrow \mathcal{C}$ on objects by $K(A) = D$ and on morphisms by $K(f) = m_i$, where $p(f) = p_i$. Let us show that K is indeed a functor. First, note that $K(\text{id}_A) = \text{id}_D = \text{id}_{K(A)}$: let $p(\text{id}_A) = p_i$; since $p_i = p(\text{id}_A) = \text{id}_{j_A(A)}$ coherence requirements force that $m_i = \text{id}_D$. Then, let us show that $K(g \circ f) = K(g) \circ K(f)$, where $f : A \rightarrow B$ and $g : B \rightarrow C$. Let k and l be positive integers such that $p(f) = p_k = j_B \circ f \circ j_A^{-1}$ and $p(g) = p_l = j_C \circ g \circ j_B^{-1}$. Let s be an integer such that $p_s = j_C \circ g \circ f \circ j_A^{-1}$. Then $p_l \circ p_k = p_s$, so the coherence requirements imply that $m_l \circ m_k = m_s$. Finally, $K(g \circ f) = m_s = m_l \circ m_k = K(g) \circ K(f)$. The coherence requirements also ensure that K preserves embeddings.

Let us now show that the set of arrows $\eta_A = e \circ j_A$ constitutes a natural transformation $\eta : \text{ID} \rightarrow K$. Take any \mathcal{A} -morphism $f : A \rightarrow B$. Then $p(f) = p_i = j_B \circ f \circ j_A^{-1}$ is a partial morphism of L whose extension is m_i . This is why the following diagram commutes (where the dashed arrow indicates a partial morphism):

$$\begin{array}{ccccc} A & \xrightarrow{j_A} & L & \xrightarrow{e} & D \\ \downarrow f & & \downarrow p_i & & \downarrow m_i = K(f) \\ B & \xrightarrow{j_B} & L & \xrightarrow{e} & D \end{array}$$

η_A (top curved arrow), η_B (bottom curved arrow)

Finally, let us show that $K(A)$ embeds all one-point extensions of A . Let $A \hookrightarrow B$. Then there is an $h : B \hookrightarrow L$ such that

$$\begin{array}{ccc} A & \xrightarrow{j_A} & L \\ \downarrow \cdot & \nearrow h & \\ B & & \end{array}$$

since L is the Fraïssé limit of \mathcal{A} . Therefore,

$$\begin{array}{ccccc} & & \eta_A & & \\ & \hookrightarrow & & \hookrightarrow & \\ A & \xrightarrow{j_A} & L & \xrightarrow{e} & D = K(A) \\ \downarrow \cdot & \nearrow h & & \nearrow eoh & \\ B & & & & \end{array}$$

which concludes the proof. \square

Note that the Henson graph H_n , $n \geq 3$, does not have the morphism extension property with respect to all graph homomorphisms (for otherwise there would be a Katětov functor defined on the category of all finite K_n -free graphs and all graph homomorphisms, and we know that such a functor cannot exist).

Conjecture. Every Fraïssé limit has the morphism extension property with respect to embeddings.

The following theorem shows that the existence of a Katětov functor for varieties of algebras understood as categories whose objects are the algebras of the variety and morphisms are embeddings is equivalent to the amalgamation property for the category of finitely generated algebras of the variety.

Theorem 3.7 *Let Δ be an algebraic language and let \mathcal{V} be a variety of Δ -algebras understood as a category whose objects are Δ -algebras and morphisms are embeddings. Let \mathcal{A} be the full subcategory of \mathcal{V} spanned by all finitely generated algebras in \mathcal{V} and let \mathcal{C} be the full subcategory of \mathcal{V} spanned by all countably generated algebras in \mathcal{V} . Assume additionally that there are only countably many isomorphism types in \mathcal{A} . Then there exists a Katětov functor $K : \mathcal{A} \rightarrow \mathcal{C}$ if and only if \mathcal{A} is an amalgamation class.*

Proof. (\Rightarrow) Immediately from Theorem 3.3.

(\Leftarrow) Recall that a *partial algebra* consists of a set A and some partial operations on A , where a partial operation is any partial mapping $A^n \rightarrow A$ for some n (see [9] for further reference on partial algebras). Clearly, the class of all partial algebras of

a fixed type is a free amalgamation class because we can simply identify the elements of the common subalgebra and leave everything else undefined.

According to Theorem 2.15 it suffices to show that \mathcal{A} has one-point extension pushouts in \mathcal{C} . Take any $A_0, A_1, A_2 \in \text{Ob}(\mathcal{A})$ such that A_0 embeds into A_1 and A_2 is a one-point extension of A_0 . Without loss of generality we may assume that $A_0 \leq A_1$ and $A_0 \leq A_2$. Let $G \subseteq A_0$ be a finite set which generates A_0 , choose $x \in A_2 \setminus A_0$ so that $G \cup \{x\}$ generates A_2 and let H be a finite set disjoint from G such that $G \cup H$ generates A_1 . Let $S = A_1 \oplus_{A_0} A_2$ be the partial algebra which arises as the free amalgam of A_1 and A_2 over A_0 in the class of all partial Δ -algebras. Since \mathcal{A} has the amalgamation property, there is a $C \in \text{Ob}(\mathcal{A})$ such that

$$\begin{array}{ccc} A_0 & \xrightarrow{\cdot} & A_2 \\ \downarrow \leq & & \downarrow \\ A_1 & \xrightarrow{\cdot} & C \end{array}$$

whence follows that C embeds the partial algebra S in the sense of [9, §28]. It is a well-known fact (see again [9, §28]) that if P is a partial algebra which embeds into some total algebra from \mathcal{V} then the free algebra $\mathbb{F}_{\mathcal{V}}(P)$ exists in \mathcal{V} . Therefore, $\mathbb{F}_{\mathcal{V}}(S)$ exists and belongs to \mathcal{V} . It is easy to see that $\mathbb{F}_{\mathcal{V}}(S)$ is generated by $\{x\} \cup G \cup H$, so $\mathbb{F}_{\mathcal{V}}(S)$ is a one-point extension of A_1 . It clearly embeds A_2 , so we have that

$$\begin{array}{ccc} A_0 & \xrightarrow{\cdot} & A_2 \\ \downarrow \leq & & \downarrow \\ A_1 & \xrightarrow{\cdot} & \mathbb{F}_{\mathcal{V}}(S) \end{array}$$

The universal mapping property, which is the defining property of free algebras, ensures that the above commuting square is actually a pushout square in \mathcal{C} . This completes the proof that \mathcal{A} has one-point extension pushouts in \mathcal{C} . \square

Corollary 3.8 *A Katětov functor exists for the category of all finite semilattices, the category of all finite lattices and for the category of all finite Boolean algebras.*

Proof. The proof follows immediately from the fact that all the three classes of algebras are well-known examples of amalgamation classes. \square

3.2 Automorphism groups and endomorphism monoids

The existence of a Katětov functor enables us to quickly conclude that the automorphism group of the corresponding Fraïssé limit is universal, as is the monoid of \mathcal{C} -endomorphisms. As an immediate consequence of Theorem 3.3 we have:

Corollary 3.9 *Let $K : \mathcal{A} \rightarrow \mathcal{C}$ be a Katětov functor and let L be the Fraïssé limit of \mathcal{A} (which exists by Theorem 3.3). Then for every $C \in \text{Ob}(\mathcal{C})$:*

- $\text{Aut}(C) \hookrightarrow \text{Aut}(L)$;
- $\text{End}_{\mathcal{C}}(C) \hookrightarrow \text{End}_{\mathcal{C}}(L)$.

Proof. Since K^ω is a functor, we immediately get that $\text{Aut}(C) \hookrightarrow \text{Aut}(K^\omega(C))$ via $f \mapsto K^\omega(f)$ and that $\text{End}_{\mathcal{C}}(C) \hookrightarrow \text{End}_{\mathcal{C}}(K^\omega(C))$ via $f \mapsto K^\omega(f)$. But, $K^\omega(C) \cong L$ due to Theorem 3.3. \square

Recall that $\text{End}_{\mathcal{C}}(X)$ may be just the set of all embeddings of X into itself, in case other homomorphisms are not in \mathcal{C} . This is the case, for example, in the class of K_n -free graphs, where there is no Katětov functor acting on all homomorphisms.

Corollary 3.10 *For the following Fraïssé limits L we have that $\text{Aut}(L)$ embeds all permutation groups on a countable set:*

- the random graph (proved originally in [10]),
- Henson graphs (proved originally in [10]),
- the random digraph,
- the rational Urysohn space (follows also from [20]),
- the random poset,
- the countable atomless Boolean algebra,
- the random semilattice,
- the random lattice,

For the following Fraïssé limits L we have that $\text{End}(L)$ embeds all transformation monoids on a countable set:

- the random graph (proved originally in [4]),
- the random digraph,
- the rational Urysohn space,
- the random poset (proved originally in [7]),
- the countable atomless Boolean algebra.

Proof. Having in mind Corollary 3.9, in each case it suffices to show that the corresponding category \mathcal{C} contains a countable structure whose automorphism group embeds $\text{Sym}(\mathbb{N})$ and whose endomorphism monoid embeds $\mathbb{N}^{\mathbb{N}}$ considered as a transformation monoid. For example, in case of the rational Urysohn space it suffices to consider the metric space (\mathbb{N}, d) where $d(m, n) = 1$ for all $m, n \in \mathbb{N}$, while in the case of the random Boolean algebra it suffices to consider the free Boolean algebra on \aleph_0 generators. \square

For some applications it is important to know whether the embeddings mentioned in Corollary 3.9 above are topological embeddings, when $\text{Aut}(X)$ and $\text{End}(X)$ are endowed with the *pointwise topology*, that is, the topology inherited from the power X^X , where X carries the discrete topology. This natural topology makes the composition operation (and the inverse, in case of $\text{Aut}(X)$) continuous. Note that $\text{Aut}(X) \subseteq \text{End}(X)$ are closed in X^X (not being a homomorphism is witnessed by a finite set). In case where X is countable, X^X is the well-known *Baire space*, a canonical Polish space, and $\text{Aut}(X)$ is isomorphic to a closed subgroup of the countable infinite symmetric group S_{∞} . The importance of such groups is demonstrated in the pioneering work [13] connecting Fraïssé theory with general Ramsey theory and topological dynamics. As we shall see in a moment, *every* Katětov functor embeds hom-sets preserving their pointwise topology.

Given \mathcal{C} -objects X, Y , denote by $\mathcal{C}(X, Y)$ the set of all \mathcal{C} -morphisms from X to Y , endowed with the pointwise topology, that is, the topology inherited from the product X^Y with X discrete. Note that a sequence $f_n \in \mathcal{C}(X, Y)$ converges to $f \in \mathcal{C}(X, Y)$ if and only if for every finite set $S \subseteq X$ there is n_0 such that $f_n \upharpoonright S = f \upharpoonright S$ for every $n \geq n_0$.

Proposition 3.11 *Let $K: \mathcal{C} \rightarrow \mathcal{C}$ be a Katětov functor. Then for every \mathcal{C} -objects X, Y , the mapping*

$$\mathcal{C}(X, Y) \ni f \mapsto K(f) \in \mathcal{C}(K(X), K(Y))$$

is a topological embedding.

Proof. From the definition of a Katětov functor, we know that the mapping above (which we also denote by K) is one-to-one, as $K(f)$ can be viewed as an extension of f (the natural transformation η consists of embeddings). Let f_n be a sequence in $\mathcal{C}(X, Y)$. If $K(f_n) \rightarrow K(f)$ pointwise, then $f_n \rightarrow f$ pointwise, due to the remark above. Now suppose $f_n \rightarrow f$ pointwise and fix $a \in K(X)$. Choose a finite $S \subseteq X$ such that the structure $A = \langle S \rangle$ generated by S has the property that $a \in K(A)$, after identifying $K(A)$ with a suitable substructure of $K(X)$ (recall that $K(X)$ is the standard colimit of $K(F)$, where F runs over all finitely generated substructures of X). There is n_0 such that $f_n \upharpoonright S = f \upharpoonright S$ whenever $n \geq n_0$. Then also $f_n \upharpoonright A = f \upharpoonright A$ for every $n \geq n_0$. Hence $K(f_n) \upharpoonright K(A) = K(f) \upharpoonright K(A)$ whenever $n \geq n_0$, showing that $f_n(a) \rightarrow f(a)$ in the discrete topology. Finally, note that the topology on

$\mathcal{C}(X, Y)$ is always metrizable (and therefore determined by sequences), because X is countably generated and $\mathcal{C}(X, Y)$ is homeomorphic (via the restriction operator) to a subspace of Y^G , where G is a countable set generating X . \square

Corollary 3.12 *The embeddings appearing in Corollary 3.9 are topological with respect to the pointwise topology.*

4 Semigroup Bergman property

Following [16], we say that a semigroup S is *semigroup Cayley bounded with respect to a generating set U* if $S = U \cup U^2 \cup \dots \cup U^n$ for some $n \in \mathbb{N}$. We say that a semigroup S has the *semigroup Bergman property* if it is semigroup Cayley bounded with respect to every generating set.

A semigroup S has *Sierpiński rank n* if n is the least positive integer such that for any countable $T \subseteq S$ there exist $s_1, \dots, s_n \in S$ such that $T \subseteq \langle s_1, \dots, s_n \rangle$. If no such n exists, the Sierpiński rank of S is said to be infinite. A semigroup S is *strongly distorted* if there exists a sequence of natural numbers l_1, l_2, l_3, \dots and an $N \in \mathbb{N}$ such that for any sequence $a_1, a_2, a_3, \dots \in S$ there exist $s_1, \dots, s_N \in S$ and a sequence of words w_1, w_2, w_3, \dots over the alphabet $\{x_1, x_2, \dots, x_N\}$ such that $|w_n| \leq l_n$ and $a_n = w_n(s_1, \dots, s_N)$ for all n .

Lemma 4.1 ([16]) *If S is a strongly distorted semigroup which is not finitely generated, then S has the Bergman property.*

It was shown in [17] that $\text{End}(R)$, the endomorphism monoid of the random graph, is strongly distorted and hence has the semigroup Bergman property since it is not finitely generated. The idea from [17] was later in [6] directly generalized to classes of structures with coproducts. Here, we present a general treatment in the context of classes for which a Katětov functor exists, and where the (JEP) can be carried out constructively in the sense of the following definition.

Definition 4.2 A category \mathcal{C} has *natural (JEP)* if there exists a covariant functor $F : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$ such that

- for all $C, D \in \text{Ob}(\mathcal{C})$ there exist embeddings $\lambda_C : C \hookrightarrow F(C, D)$ and $\rho_D : D \hookrightarrow F(C, D)$, and
- for every pair of morphisms $f : C \rightarrow C'$ and $g : D \rightarrow D'$ the diagram below commutes:

$$\begin{array}{ccccc} C & \xrightarrow{\lambda_C} & F(C, D) & \xleftarrow{\rho_D} & D \\ f \downarrow & & \downarrow F(f, g) & & \downarrow g \\ C' & \xrightarrow{\lambda_{C'}} & F(C', D') & \xleftarrow{\rho_{D'}} & D' \end{array}$$

We also say that F is a *natural (JEP) functor* for \mathcal{C} .

A category \mathcal{C} has *retractive natural (JEP)* if \mathcal{C} has natural (JEP) and the functor F has the following additional property: for every $C \in \text{Ob}(\mathcal{C})$ there exist morphisms $\rho_C^*, \lambda_C^* : F(C, C) \rightarrow C$ such that $\rho_C^* \circ \rho_C = \text{id}_C = \lambda_C^* \circ \lambda_C$.

Remark 4.3 Note that since F is a covariant functor, the following also holds:

- $F(\text{id}_C, \text{id}_D) = \text{id}_{F(C, D)}$ for all $C, D \in \text{Ob}(\mathcal{C})$,
- for all $f_1 : B_1 \rightarrow C_1, f_2 : B_2 \rightarrow C_2, g_1 : C_1 \rightarrow D_1, g_2 : C_2 \rightarrow D_2$ we have $F(g_1 \circ f_1, g_2 \circ f_2) = F(g_1, g_2) \circ F(f_1, f_2)$, and

$$\begin{array}{ccccc}
 \bullet & A \xrightarrow{f_3} C & \text{and} & P \xrightarrow{g_3} Q & \text{implies} & F(A, P) \xrightarrow{F(f_3, g_3)} F(C, Q) \\
 & \downarrow f_1 & & \downarrow g_1 & & \downarrow F(f_1, g_1) \\
 & B \xrightarrow{f_4} D & & R \xrightarrow{g_4} S & & F(B, R) \xrightarrow{F(f_4, g_4)} F(D, S) \\
 & & & & & \downarrow F(f_2, g_2)
 \end{array}$$

Example 4.4 Any category with coproducts (such as the category of graphs, posets, digraphs) has retractive natural (JEP): just take $F(C, D)$ to be the coproduct of C and D .

Example 4.5 The category of all countable metric spaces with distances in $[0, 1]_{\mathbb{Q}} = \mathbb{Q} \cap [0, 1]$ and nonexpansive mappings has retractive natural (JEP): take $F(C, D)$ to be the disjoint union of C and D where the distance between any point in C and any point in D is 1.

On the other hand, it is easy to show that the category of all countable metric spaces with distances in \mathbb{Q} and nonexpansive mappings does not have natural (JEP). Suppose, to the contrary, that there exists a functor F which realizes the natural (JEP) in this category, let U be the rational Urysohn space and let $W = F(U, U)$. Let $a_0, b_0 \in U$ be arbitrary but fixed, and let $\delta = d_W(\lambda_U(a_0), \rho_U(b_0))$. Take any $a, b \in U$, let $c_a : U \rightarrow U : x \mapsto a$ and $c_b : U \rightarrow U : x \mapsto b$ be the constant maps and put $\Phi = F(c_a, c_b)$. Then $d_W(\lambda_U(a), \rho_U(b)) = d_W(\lambda_U(c_a(a_0)), \rho_U(c_b(b_0))) = d_W(\Phi(\lambda_U(a_0)), \Phi(\rho_U(b_0))) \leq d_W(\lambda_U(a_0), \rho_U(b_0)) = \delta$, because Φ is nonexpansive. Now, for $a_1, a_2 \in U$ we have $d_U(a_1, a_2) = d_W(\lambda_U(a_1), \lambda_U(a_2)) \leq d_W(\lambda_U(a_1), \rho_U(b)) + d_W(\lambda_U(a_2), \rho_U(b)) \leq 2\delta$. Hence, $\text{diam}(U) \leq 2\delta$. Contradiction.

Example 4.6 Let Δ be the language consisting of function symbols and constant symbols only so that Δ -structures are actually Δ -algebras, and assume that Δ contains a constant symbol 1. Then the category of Δ -algebras has retractive natural (JEP): take $F(C, D)$ to be $C \times D$ where $\lambda_C : c \mapsto \langle c, 1^D \rangle$, $\rho_D : d \mapsto \langle 1^C, d \rangle$, $\lambda_C^* = \pi_1$ and $\rho_D^* = \pi_2$.

Our aim in this section is to prove the following theorem:

Theorem 4.7 Assume that there exists a Katětov functor $K : \mathcal{A} \rightarrow \mathcal{C}$ and assume that \mathcal{C} has retractive natural (JEP). Let L be the Fraïssé limit of \mathcal{A} (which exists by Theorem 3.3). Assume additionally that there is a retraction $r : K(L) \rightarrow L$ such that $r \circ \eta_L = \text{id}_L$. Then $\text{End}_{\mathcal{C}}(L)$ is strongly distorted and its Sierpiński rank is at most 5. Consequently, if $\text{End}_{\mathcal{C}}(L)$ is not finitely generated then it has the Bergman property.

The proof of the theorem requires some technical prerequisites. Let us denote the functor which realizes (JEP) in \mathcal{C} by (\cdot, \cdot) so that (C, D) denotes its action on objects, and (f, g) its action on morphisms. For objects $C_1, C_2, C_3, \dots, C_n$ and morphisms $f, g, f_1, f_2, f_3, \dots, f_n$ of \mathcal{C} let

$$\begin{aligned} [C_1, C_2, C_3, \dots, C_n] &= (((C_1, C_2), C_3), \dots), C_n), \\ [f_1, f_2, f_3, \dots, f_n] &= (((f_1, f_2), f_3), \dots), f_n), \\ [f, g]_n &= [f, \underbrace{g, \dots, g}_n], \text{ with } [f, g]_0 = f. \end{aligned}$$

Moreover, let

$$\begin{aligned} L_1 &= L, \\ L_n &= (L_{n-1}, L) = \underbrace{[L, L, \dots, L]}_n, \text{ for } n \geq 2. \end{aligned}$$

Let C denote the colimit of the following chain in \mathcal{C} with the canonical embeddings denoted by ι_n :

$$\begin{array}{ccccccc} L_1 & \xrightarrow{\lambda_{L_1}} & L_2 & \xrightarrow{\lambda_{L_2}} & L_3 & \xrightarrow{\lambda_{L_3}} & \dots \\ & \searrow \iota_1 & & \searrow \iota_2 & & \searrow \iota_3 & \\ & & & & & & C \end{array}$$

Let L be the Fraïssé limit of \mathcal{A} , which exists by Theorem 3.3. We know that $K^\omega(C) \cong L$, so let us fix an isomorphism

$$\alpha : K^\omega(C) \xrightarrow{\cong} L.$$

The following diagram commutes because (\cdot, \cdot) is a natural (JEP) functor:

$$\begin{array}{ccccccc} L_1 & \xrightarrow{\lambda_{L_1}} & L_2 & \xrightarrow{\lambda_{L_2}} & L_3 & \xrightarrow{\lambda_{L_3}} & L_4 \xrightarrow{\lambda_{L_4}} \dots \\ \searrow \text{id}_L & \lambda_L^* \downarrow & & [\lambda_L^*, \text{id}_L]_1 \downarrow & & [\lambda_L^*, \text{id}_L]_2 \downarrow & \\ & L_1 & \xrightarrow{\lambda_{L_1}} & L_2 & \xrightarrow{\lambda_{L_2}} & L_3 & \xrightarrow{\lambda_{L_3}} \dots \end{array}$$

so the following diagram also commutes:

$$\begin{array}{ccccccc}
L_1 & \xrightarrow{\lambda_{L_1}} & L_2 & \xrightarrow{\lambda_{L_2}} & L_3 & \xrightarrow{\lambda_{L_3}} & L_4 \xrightarrow{\lambda_{L_4}} \dots \\
& \searrow \text{id}_L & \downarrow \lambda_L^* & & \downarrow [\lambda_L^*, \text{id}_L]_1 & & \downarrow [\lambda_L^*, \text{id}_L]_2 \\
& & L_1 & \xleftarrow{\lambda_{L_1}^*} & L_2 & \xleftarrow{\lambda_{L_2}^*} & L_3 \xleftarrow{\lambda_{L_3}^*} \dots
\end{array}$$

Therefore, there is a compatible cone with the tip at L and the morphisms id_L , λ_L^* , $\lambda_{L_1}^* \circ [\lambda_L^*, \text{id}_L]_1$, $\lambda_{L_1}^* \circ \lambda_{L_2}^* \circ [\lambda_L^*, \text{id}_L]_2 \dots$ over the chain $L_1 \hookrightarrow L_2 \hookrightarrow L_3 \hookrightarrow \dots$. Since C is a colimit of the chain, there is a unique $\beta : C \rightarrow L$ such that

$$\begin{array}{ccc}
C & \xrightarrow{\beta} & L \\
\uparrow \iota_1 & \swarrow \iota_2 & \searrow \text{id}_L \\
L_1 & \xrightarrow{\lambda_{L_1}} & L_2 \xrightarrow{\lambda_{L_2}} L_3 \xrightarrow{\lambda_{L_3}} \dots
\end{array}$$

(Note: The diagram also includes diagonal arrows from C to L_2 labeled ι_3 and from L_3 to L labeled $\lambda_L^* \circ \lambda_{L_1}^* \circ [\lambda_L^*, \text{id}_L]_1$.)

In particular,

$$\beta \circ \iota_1 = \text{id}_L. \quad (6)$$

As the next step in the construction, note that the following diagram commutes (again due to the fact that (\cdot, \cdot) is a natural (JEP) functor):

$$\begin{array}{ccccccc}
L_1 & \xrightarrow{\lambda_{L_1}} & L_2 & \xrightarrow{\lambda_{L_2}} & L_3 & \xrightarrow{\lambda_{L_3}} & L_4 \xrightarrow{\lambda_{L_4}} \dots \\
\rho_L \downarrow & & \downarrow [\rho_L, \text{id}_L]_1 & & \downarrow [\rho_L, \text{id}_L]_2 & & \downarrow [\rho_L, \text{id}_L]_3 \\
L_2 & \xrightarrow{\lambda_{L_2}} & L_3 & \xrightarrow{\lambda_{L_3}} & L_4 & \xrightarrow{\lambda_{L_4}} & L_5 \xrightarrow{\lambda_{L_5}} \dots \\
\rho_L^* \downarrow & & \downarrow [\rho_L^*, \text{id}_L]_1 & & \downarrow [\rho_L^*, \text{id}_L]_2 & & \downarrow [\rho_L^*, \text{id}_L]_3 \\
L_1 & \xrightarrow{\lambda_{L_1}} & L_2 & \xrightarrow{\lambda_{L_2}} & L_3 & \xrightarrow{\lambda_{L_3}} & L_4 \xrightarrow{\lambda_{L_4}} \dots
\end{array}$$

Therefore, there is a compatible cone with the tip at C and the morphisms $\iota_2 \circ \rho_L$, $\iota_3 \circ [\rho_L, \text{id}_L]_1$, $\iota_4 \circ [\rho_L, \text{id}_L]_2 \dots$ over the chain $L_1 \hookrightarrow L_2 \hookrightarrow L_3 \hookrightarrow \dots$. Since C is a colimit of the chain, there is a unique $\sigma : C \rightarrow C$ such that

$$\begin{array}{ccc}
C & \xrightarrow{\sigma} & C \\
\uparrow \iota_1 & \swarrow \iota_2 & \searrow \iota_2 \circ \rho_L \\
L_1 & \xrightarrow{\lambda_{L_1}} & L_2 \xrightarrow{\lambda_{L_2}} L_3 \xrightarrow{\lambda_{L_3}} \dots
\end{array}$$

(Note: The diagram also includes diagonal arrows from C to L_3 labeled $\iota_3 \circ [\rho_L, \text{id}_L]_1$ and from L_4 to C labeled $\iota_4 \circ [\rho_L, \text{id}_L]_2$.)

or, explicitly,

$$\sigma \circ \iota_n = \iota_{n+1} \circ [\rho_L, \text{id}_L]_{n-1}, \text{ for all } n \geq 1.$$

An easy induction on n then suffices to show that

$$\sigma^n \circ \iota_1 = \iota_{n+1} \circ [\rho_L, \text{id}_L]_{n-1} \circ \dots \circ [\rho_L, \text{id}_L]_1 \circ \rho_L, \text{ for all } n \geq 1. \quad (7)$$

Also, there is a compatible cone with the tip at C and the morphisms $\iota_1 \circ \rho_L^*$, $\iota_2 \circ [\rho_L^*, \text{id}_L]_1$, $\iota_3 \circ [\rho_L^*, \text{id}_L]_2 \dots$ over the chain $L_2 \hookrightarrow L_3 \hookrightarrow L_4 \hookrightarrow \dots$, so there is a unique $\tau : C \rightarrow C$ such that

or, explicitly,

$$\tau \circ \iota_{n+1} = \iota_n \circ [\rho_L^*, \text{id}_L]_{n-1}, \text{ for all } n \geq 1.$$

Another easy induction on n suffices to show that

$$\tau^n \circ \iota_{n+1} = \iota_1 \circ \rho_L^* \circ [\rho_L^*, \text{id}_L]_1 \circ \dots \circ [\rho_L^*, \text{id}_L]_{n-1}, \text{ for all } n \geq 1. \quad (8)$$

Let $\bar{f} = (f_1, f_2, \dots)$ be a sequence of \mathcal{C} -endomorphisms of L . As the final step, we shall now construct an endomorphism $\varphi(\bar{f}) : C \rightarrow C$ which encodes the sequence \bar{f} . Using once more the fact that (\cdot, \cdot) is a natural (JEP) functor, we immediately get that the following diagram commutes:

so there is a unique $\varphi(\bar{f}) : C \rightarrow C$ such that

or, explicitly,

$$\varphi(\bar{f}) \circ \iota_n = \iota_n \circ [f_1, f_2, \dots, f_n], \text{ for all } n \geq 1.$$

Lemma 4.8 (a) $\varphi(\bar{f}) \circ \iota_1 = \iota_1 \circ f_1$;

(b) $\varphi(\bar{f}) \circ \iota_2 \circ \rho_L = \iota_1 \circ \rho_L \circ f_2$;

- (c) $\varphi(\bar{f}) \circ \iota_n \circ [\rho_L, \text{id}_L]_{n-2} \circ \dots \circ [\rho_L, \text{id}_L]_1 \circ \rho_L = \iota_n \circ [\rho_L, \text{id}_L]_{n-2} \circ \dots \circ [\rho_L, \text{id}_L]_1 \circ \rho_L \circ f_n$,
for all $n \geq 3$.

Proof. (a) This is immediate from the construction of $\varphi(\bar{f})$.

(b) It suffices to note that the diagram below commutes. The square on the left commutes because (\cdot, \cdot) is natural, while the square on the right commutes by the construction of $\varphi(\bar{f})$.

$$\begin{array}{ccccc} L_1 & \xrightarrow{\rho_L} & L_2 & \xrightarrow{\iota_2} & C \\ f_2 \downarrow & & [f_1, f_2] \downarrow & & \downarrow \varphi(\bar{f}) \\ L_1 & \xrightarrow{\rho_L} & L_2 & \xrightarrow{\iota_2} & C \end{array}$$

(c) This follows by induction on n . Just to illustrate the main ideas (which are straightforward, anyhow) we show the case $n = 4$. The following diagram commutes:

$$\begin{array}{ccccccccc} L_1 & \xrightarrow{\rho_L} & L_2 & \xrightarrow{[\rho_L, \text{id}_L]_1} & L_3 & \xrightarrow{[\rho_L, \text{id}_L]_2} & L_4 & \xrightarrow{\iota_4} & C \\ f_4 \downarrow & & [f_3, f_4] \downarrow & & [f_2, f_3, f_4] \downarrow & & [f_1, f_2, f_3, f_4] \downarrow & & \downarrow \varphi \\ L_1 & \xrightarrow{\rho_L} & L_2 & \xrightarrow{[\rho_L, \text{id}_L]_1} & L_3 & \xrightarrow{[\rho_L, \text{id}_L]_2} & L_4 & \xrightarrow{\iota_4} & C \end{array}$$

The leftmost square commutes because (\cdot, \cdot) is natural, while the rightmost square commutes by the construction of $\varphi(\bar{f})$. To see that the second square in this row commutes, just apply the functor (\cdot, \cdot) to the following two commutative squares (see Remark 4.3):

$$\begin{array}{ccc} L_1 & \xrightarrow{\rho_L} & L_2 \\ f_3 \downarrow & & \downarrow [f_2, f_3] \\ L_1 & \xrightarrow{\rho_L} & L_2 \end{array} \quad \begin{array}{ccc} L & \xrightarrow{\text{id}_L} & L \\ f_4 \downarrow & & \downarrow f_4 \\ L & \xrightarrow{\text{id}_L} & L \end{array}$$

The same argument suffices to show that the third square in the row commutes too. \square

Lemma 4.9 (a) $\beta \circ \varphi(\bar{f}) \circ \iota_1 = f_1$;

(b) $\beta \circ \tau^n \circ \varphi(\bar{f}) \circ \sigma^n \circ \iota_1 = f_{n+1}$.

Proof. In order to make it easier to follow the calculations we underline the expression that is to be reduced in the following step.

(a) $\beta \circ \underline{\varphi(\bar{f})} \circ \underline{\iota_1} = \underline{\beta \circ \iota_1} \circ f_1 = f_1$, by Lemma 4.8 and (6).

$$(b) \beta \circ \tau^n \circ \varphi(\bar{f}) \circ (\sigma^n \circ \iota_1) =$$

$$\begin{aligned} [\text{by (7)}] &= \beta \circ \tau^n \circ \varphi(\bar{f}) \circ \iota_{n+1} \circ [\rho_L, \text{id}_L]_{n-1} \circ \dots \circ [\rho_L, \text{id}_L]_1 \circ \rho_L \\ [\text{Lemma 4.8}] &= \beta \circ \tau^n \circ \iota_{n+1} \circ [\rho_L, \text{id}_L]_{n-1} \circ \dots \circ [\rho_L, \text{id}_L]_1 \circ \rho_L \circ f_{n+1} \\ [\text{by (8)}] &= \underline{\beta \circ \iota_1} \circ \rho_L^* \circ [\rho_L^*, \text{id}_L]_1 \circ \dots \circ [\rho_L^*, \text{id}_L]_{n-1} \circ \\ &\quad \circ [\rho_L, \text{id}_L]_{n-1} \circ \dots \circ [\rho_L, \text{id}_L]_1 \circ \rho_L \circ f_{n+1} \\ [\text{by (6)}] &= \rho_L^* \circ [\rho_L^*, \text{id}_L]_1 \circ \dots \circ [\rho_L^*, \text{id}_L]_{n-1} \circ \underline{[\rho_L, \text{id}_L]_{n-1} \circ} \\ &\quad \circ [\rho_L, \text{id}_L]_{n-2} \circ \dots \circ [\rho_L, \text{id}_L]_1 \circ \rho_L \circ f_{n+1} \\ &= \dots = f_{n+1}, \end{aligned}$$

since $[\rho_L^*, \text{id}_L]_j \circ [\rho_L, \text{id}_L]_j = \text{id}_L$, for all j . \square

We are now ready to prove Theorem 4.7.

Proof. (of Theorem 4.7) We are going to show that $\text{End}(K^\omega(C))$, which is isomorphic to $\text{End}(L)$ because $L \cong K^\omega(C)$, is strongly distorted and that the Sierpiński rank of $\text{End}(K^\omega(C))$ is at most 5. Take any countable sequence $f_1, f_2, \dots \in \text{End}(K^\omega(C))$, and let us construct $\tilde{\alpha}, \tilde{\beta}, \tilde{\sigma}, \tilde{\tau}, \tilde{\varphi} \in \text{End}(K^\omega(C))$ as follows, with the notation introduced above.

Let $\tilde{\alpha} = \eta_C^\omega \circ \iota_1 \circ \alpha : K^\omega(C) \rightarrow K^\omega(C)$. We shall construct $\tilde{\beta} : K^\omega(C) \rightarrow K^\omega(C)$ such that $\tilde{\beta} \circ \eta_C^\omega = \alpha^{-1} \circ \beta$. Since η is natural, the diagram on the left below commutes, so by taking $\beta_1 = r \circ K(\beta)$ we have that the diagram on the right also commutes:

$$\begin{array}{ccc} C & \xrightarrow{\beta} & L \\ \eta_C \downarrow & & \downarrow \eta_L \\ K(C) & \xrightarrow{K(\beta)} & K(L) \end{array} \quad \begin{array}{ccc} C & \xrightarrow{\beta} & L \\ \eta_C \downarrow & \nearrow \beta_1 & \\ K(C) & & \end{array}$$

Analogously, the following diagrams also commute where $\beta_2 = r \circ K(\beta_1)$:

$$\begin{array}{ccc} K(C) & \xrightarrow{\beta_1} & L \\ \eta_{K(C)} \downarrow & & \downarrow \eta_L \\ K^2(C) & \xrightarrow{K(\beta_1)} & K(L) \end{array} \quad \begin{array}{ccc} K(C) & \xrightarrow{\beta_1} & L \\ \eta_{K(C)} \downarrow & \nearrow \beta_2 & \\ K^2(C) & & \end{array}$$

And so on. We get a sequence of morphisms $\beta_n : K^n(C) \rightarrow L$ such that

$$\begin{array}{ccc} K^n(C) & \xrightarrow{\beta_n} & L \\ \eta_{K^n(C)} \downarrow & \nearrow \beta_{n+1} & \\ K^{n+1}(C) & & \end{array}$$

Since $K^\omega(C)$ is the colimit of the chain

$$C \xrightarrow{\eta_C} K(C) \xrightarrow{\eta_{K(C)}} K^2(C) \xrightarrow{\eta_{K^2(C)}} K^3(C) \hookrightarrow \dots$$

there is a unique mediating morphism $\beta_\omega : K^\omega(C) \rightarrow L$ such that

$$\begin{array}{ccccc} & & \beta_\omega & & \\ & \nearrow & & \searrow & \\ K^\omega(C) & \xleftarrow{\quad} & K^n(C) & \xrightarrow{\beta_n} & L \\ & \nwarrow & \downarrow \eta_{K^n(C)} & \nearrow \beta_{n+1} & \\ & & K^{n+1}(C) & & \end{array}$$

In particular, $\beta_\omega \circ \eta_C^\omega = \beta$. Now put $\tilde{\beta} = \alpha^{-1} \circ \beta_\omega$.

Finally, let $\tilde{\sigma} = K^\omega(\sigma)$ and $\tilde{\tau} = K^\omega(\tau)$, let $f_n^\alpha = \alpha \circ f_n \circ \alpha^{-1}$, and let $\tilde{\varphi} = K^\omega(\varphi(\bar{g}))$ where $\bar{g} = (f_1^\alpha, f_2^\alpha, \dots)$. Then

$$\begin{aligned} \tilde{\beta} \circ \tilde{\varphi} \circ \tilde{\alpha} &= \tilde{\beta} \circ \underline{K^\omega(\varphi(\bar{g})) \circ \eta_C^\omega} \circ \iota_1 \circ \alpha \\ [\eta^\omega \text{ is natural}] &= \tilde{\beta} \circ \underline{\eta_C^\omega \circ \varphi(\bar{g})} \circ \iota_1 \circ \alpha \\ [\text{definition of } \tilde{\beta}] &= \alpha^{-1} \circ \underline{\beta \circ \varphi(\bar{g})} \circ \iota_1 \circ \alpha \\ [\text{Lemma 4.9}] &= \alpha^{-1} \circ f_1^\alpha \circ \alpha = f_1, \end{aligned}$$

and

$$\begin{aligned} \tilde{\beta} \circ \tilde{\tau}^n \circ \tilde{\varphi} \circ \tilde{\sigma}^n \circ \tilde{\alpha} &= \tilde{\beta} \circ \underline{K^\omega(\tau^n \circ \varphi \circ \sigma^n) \circ \eta_C^\omega} \circ \iota_1 \circ \alpha \\ [\eta^\omega \text{ is natural}] &= \tilde{\beta} \circ \underline{\eta_C^\omega \circ \tau^n \circ \varphi \circ \sigma^n} \circ \iota_1 \circ \alpha \\ [\text{definition of } \tilde{\beta}] &= \alpha^{-1} \circ \underline{\beta \circ \tau^n \circ \varphi \circ \sigma^n} \circ \iota_1 \circ \alpha \\ [\text{Lemma 4.9}] &= \alpha^{-1} \circ f_{n+1}^\alpha \circ \alpha = f_{n+1}. \end{aligned}$$

This shows that every f_n belongs to the semigroup generated by $\tilde{\alpha}$, $\tilde{\beta}$, $\tilde{\sigma}$, $\tilde{\tau}$ and $\tilde{\varphi}$, and we uniformly have that the length of the word representing f_n is $2n + 1$. Therefore, $\text{End}(K^\omega(C))$ is strongly distorted and the Sierpiński rank of $\text{End}(K^\omega(C))$ is at most 5. Lemma 4.1 now yields that $\text{End}(L)$ has the Bergman property if it is not finitely generated. \square

Corollary 4.10 *For the following Fraïssé limits L we have that $\text{End}(L)$ has the Bergman property:*

- the random graph,
- the random digraph,

- the random poset,
- the rational Urysohn sphere (the Fraïssé limit of the category of all finite metric spaces with rational distances bounded by 1),
- the countable atomless Boolean algebra.

Proof. It is easy to see that each of the categories involved has retractive natural (JEP). In the first four cases the existence of a retraction $r : K(L) \rightarrow L$ such that $r \circ \eta_L = \text{id}_L$ where L is the corresponding Fraïssé limit follows from the explicit construction of the Katětov functor (Subsection 2.1).

Let (U, ϱ) denote the Urysohn sphere. Note that each $p \in K(U)$ is determined by a finite set $F \subseteq U$ in the sense that

$$\varrho(p, u) = \min_{x \in F} (\varrho(p, x) + \varrho(x, u)) \quad (9)$$

(see Section 5 for more details, in particular, formula (10)). Note also that enlarging the set F , the equation above remains true, because of the triangle inequality.

Suppose that $U \subseteq X_0 \subseteq K(U)$ is such that $X_0 \setminus U$ is finite and a nonexpansive retraction $r : X_0 \rightarrow U$ has already been defined. Fix $p \in K(U) \setminus X_0$. Choose a finite set F containing $r[X_0 \setminus U]$, such that (9) holds. Let $A = (X_0 \setminus U) \cup F$. Then $r \upharpoonright A : A \rightarrow U$ is a nonexpansive mapping (which is identity on F) therefore by [14, Thm. 3.18] it extends to a nonexpansive mapping on $\bar{r} : A \cup \{p\} \rightarrow U$. Let $q = \bar{r}(p)$ and let $r' = \bar{r} \cup \text{id}_U$. We claim that $r' : X_0 \cup \{p\} \rightarrow U$ is nonexpansive.

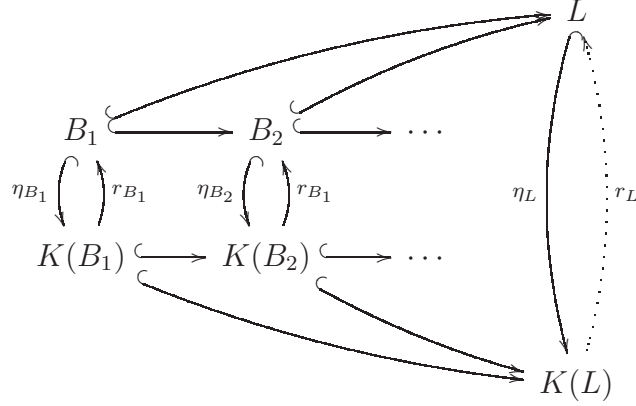
Fix $u \in U \setminus F$. Using (9), we have $\varrho(p, u) = \varrho(p, s) + \varrho(s, u)$ for some $s \in F$. Thus

$$\varrho(r'(p), r'(u)) = \varrho(\bar{r}(p), u) \leq \varrho(\bar{r}(p), s) + \varrho(s, u) \leq \varrho(p, s) + \varrho(s, u) = \varrho(p, u).$$

The last inequality follows from the fact that \bar{r} is nonexpansive on A and $\bar{r}(s) = s$. This shows that r' is a nonexpansive extension of r . Easy induction shows the existence of a nonexpansive retraction of $K(U)$ onto U .

Let us finally show that a retraction $r : K(L) \rightarrow L$ also exists in case of the category of finite Boolean algebras. Recall from Example 2.11 that for a finite Boolean algebra $B(A)$ whose set of atoms is A we have the Katětov functor $K(B(A)) = B(\{0, 1\} \times A)$ where $\eta_{B(A)} : B(A) \hookrightarrow B(\{0, 1\} \times A)$ is the unique homomorphism which takes $a \in A$ to $\langle 0, a \rangle \vee \langle 1, a \rangle \in B(\{0, 1\} \times A)$. It is now easy to see that for each finite Boolean algebra $B(A)$ there is a retraction $r_{B(A)} : K(B(A)) \rightarrow B(A)$ which takes $\langle i, a \rangle$ to a ($i \in \{0, 1\}$) and extends to the rest of $K(B(A))$ in an obvious way. Clearly, $r_{B(A)} \circ \eta_{B(A)} = \text{id}_{B(A)}$. Let L be the countable atomless Boolean algebra and let $B_1 \hookrightarrow B_2 \hookrightarrow \dots$ be a chain of finite Boolean algebras whose colimit is L . Then the colimit of the chain $K(B_1) \hookrightarrow K(B_2) \hookrightarrow \dots$ is $K(L)$ and the following diagram

commutes:



Since $K(L)$ is the colimit of the bottom chain, there is a unique mapping $r_L : K(L) \rightarrow L$ such that the diagram commutes. In particular, $r_L \circ \eta_L = \text{id}_L$. \square

5 Appendix: the original Katětov construction

For the sake of completeness we present the details of Katětov's construction in the case of finite spaces. Actually, we were unable to find any source where Katětov's extensions of metric spaces are explicitly treated as a functor acting on nonexpansive mappings.

Given a metric space X we shall denote its metric either by ϱ or by ϱ_X . Fix a finite metric space X and denote by $K(X)$ the set of all functions $\varphi : X \rightarrow [0, +\infty)$ satisfying

$$|\varphi(x_0) - \varphi(x_1)| \leq \varrho(x_0, x_1) \leq \varphi(x_0) + \varphi(x_1)$$

for every $x_0, x_1 \in X$. Elements of $K(X)$ are called *Katětov functions* on X . Given $a \in X$, the function $\hat{a}(x) = \varrho(x, a)$ is Katětov, therefore it is natural to define $\eta_X : X \rightarrow K(X)$ by $\eta_X(x) = \hat{x}$. Endow $K(X)$ with the metric

$$\varrho(\varphi, \psi) = \max_{x \in X} |\varphi(x) - \psi(x)|.$$

It is easy to see that η_X is an isometric embedding. Note that $K(X)$ is a Polish space, being a closed subspace of \mathbb{R}^X .

We now fix a nonexpansive map $f : X \rightarrow Y$ between finite metric spaces. Given $\varphi \in K(X)$, define

$$\varphi^f(y) = \min_{x \in X} (\varrho_Y(y, f(x)) + \varphi(x)). \quad (10)$$

Lemma 5.1 $\varphi^f \in K(Y)$ for every $\varphi \in K(X)$. Furthermore, given $x \in X$, we have that $\varphi^f(f(x)) \leq \varphi(x)$ and $\varphi^f(f(x)) = \varphi(x)$ whenever f is an isometric embedding.

Proof. Fix $y_0, y_1 \in Y$ and assume $\varphi^f(y_i) = \varrho_Y(y_i, f(x_i)) + \varphi(x_i)$ for $i = 0, 1$. Then

$$\begin{aligned}\varphi^f(y_0) &\leq \varrho_Y(y_0, f(x_1)) + \varphi(x_1) \\ &\leq \varrho_Y(y_0, f(x_1)) - \varrho_Y(y_1, f(x_1)) + \varrho_Y(y_1, f(x_1)) + \varphi(x_1) \\ &\leq \varrho_Y(y_0, y_1) + \varphi^f(y_1).\end{aligned}$$

Similarly, $\varphi^f(y_1) \leq \varrho_Y(y_0, y_1) + \varphi^f(y_0)$. Furthermore, using the fact that f is non-expansive and φ is Katětov, we get

$$\begin{aligned}\varrho_Y(y_0, y_1) &\leq \varrho_Y(y_0, f(x_0)) + \varrho_Y(f(x_0), f(x_1)) + \varrho_Y(y_1, f(x_1)) \\ &\leq \varrho_Y(y_0, f(x_0)) + \varrho_X(x_0, x_1) + \varrho_Y(y_1, f(x_1)) \\ &\leq \varrho_Y(y_0, f(x_0)) + \varphi(x_0) + \varphi(x_1) + \varrho_Y(y_1, f(x_1)) \\ &= \varphi^f(y_0) + \varphi^f(y_1).\end{aligned}$$

This shows that φ^f is a Katětov function. Inequality $\varphi^f(f(x)) \leq \varphi(x)$ is trivial. Finally, suppose f is an isometric embedding and fix $x \in X$. Choose $x_1 \in X$ so that $\varphi^f(f(x)) = \varrho_Y(f(x), f(x_1)) + \varphi(x_1)$. Then

$$\varphi^f(f(x)) = \varrho_X(x, x_1) + \varphi(x_1) \geq \varphi(x),$$

because φ is Katětov. This completes the proof. \square

Lemma 5.2 $\varrho(\varphi^f, \psi^f) \leq \varrho(\varphi, \psi)$ for every $\varphi, \psi \in K(X)$. Equality holds whenever f is an isometric embedding.

Proof. Fix $y \in Y$. Find $x_0 \in X$ such that $\varphi^f(y) = \varrho(y, f(x_0)) + \varphi(x_0)$. Then $\psi^f(y) \leq \varrho(y, f(x_0)) + \psi(x_0)$, therefore

$$\psi^f(y) - \varphi^f(y) \leq \psi(x_0) - \varphi(x_0) \leq |\psi(x_0) - \varphi(x_0)| \leq \varrho(\varphi, \psi).$$

By symmetry, $\varphi^f(y) - \psi^f(y) \leq \varrho(\varphi, \psi)$. Thus $|\varphi^f(y) - \psi^f(y)| \leq \varrho(\varphi, \psi)$ and hence $\varrho(\varphi^f, \psi^f) \leq \varrho(\varphi, \psi)$. Finally, if f is an isometric embedding and $\varrho(\varphi, \psi) = |\varphi(x_0) - \psi(x_0)|$ then, using Lemma 5.1, we get

$$\varrho(\varphi^f, \psi^f) \geq |\varphi^f(x) - \psi^f(x)| = |\varphi(x) - \psi(x)|$$

for every $x \in X$, which implies that $\varrho(\varphi^f, \psi^f) \geq \varrho(\varphi, \psi)$. \square

Lemma 5.3 Given nonexpansive mappings $f: X \rightarrow Y$, $g: Y \rightarrow Z$ between finite metric spaces, it holds that $\varphi^{g \circ f} = (\varphi^f)^g$ for every $\varphi \in K(X)$.

Proof. Fix $z \in Z$. We have

$$\begin{aligned}
(\varphi^f)^g(z) &= \min_{y \in Y} \left(\varrho_Z(z, g(y)) + \varphi^f(y) \right) \\
&= \min_{y \in Y, x \in X} \left(\varrho_Z(z, g(y)) + \varrho_Y(y, f(x)) + \varphi(x) \right) \\
&\geq \min_{y \in Y, x \in X} \left(\varrho_Z(z, g(y)) + \varrho_Z(g(y), gf(x)) + \varphi(x) \right) \\
&\geq \min_{x \in X} \left(\varrho_Z(z, gf(x)) + \varphi(x) \right) = \varphi^{g \circ f}(z).
\end{aligned}$$

On the other hand, using Lemma 5.1, we get

$$\begin{aligned}
(\varphi^f)^g(z) &\leq \min_{x \in X} \left(\varrho(z, gf(x)) + \varphi^f(f(x)) \right) \\
&\leq \min_{x \in X} \left(\varrho(z, gf(x)) + \varphi(x) \right) = \varphi^{g \circ f}(z).
\end{aligned}$$

□

It is obvious that $\varphi^{\text{id}_X} = \varphi$, therefore defining

$$K(f)(\varphi) = \varphi^f$$

we obtain a covariant functor K from the category of finite metric spaces into the category of Polish metric spaces, both considered with nonexpansive mappings. Furthermore, K preserves isometric embeddings (by the second part of Lemma 5.2).

Lemma 5.4 *Given a nonexpansive mapping of finite metric spaces $f: X \rightarrow Y$, the following diagram is commutative.*

$$\begin{array}{ccc}
X & \xrightarrow{\eta_X} & K(X) \\
f \downarrow & & \downarrow K(f) \\
Y & \xrightarrow{\eta_Y} & K(Y)
\end{array}$$

Proof. Fix $x \in X$. We have $(K(f) \circ \eta_X)(x) = K(f)(\widehat{x}) = \widehat{x}^f$ and $(\eta_Y \circ f)(x) = \eta_Y(f(x)) = \widehat{f(x)}$. It remains to show that $\widehat{x}^f = \widehat{f(x)}$. We have

$$\begin{aligned}
\widehat{x}^f(y) &= \min_{t \in X} \left(\varrho(y, f(t)) + \varrho(x, t) \right) \\
&\geq \min_{t \in X} \left(\varrho(y, f(t)) + \varrho(f(x), f(t)) \right) \geq \varrho(y, f(x)) = \widehat{f(x)}(y).
\end{aligned}$$

On the other hand,

$$\widehat{x}^f(y) \leq \varrho(y, f(x)) + \varrho(x, x) = \varrho(y, f(x)) = \widehat{f(x)}(y).$$

Hence $\widehat{x}^f = \widehat{f(x)}$. □

The lemma above says that η is a natural transformation from the identity functor into K . The last fact just says that K is a Katětov functor.

Proposition 5.5 *Let $e: X \rightarrow Y$ be an isometric embedding such that X is finite and $|Y \setminus X| = 1$. Then there exists an isometric embedding $g: Y \rightarrow K(X)$ such that $g \circ e = \eta_X$.*

Proof. We may assume that $Y = X \cup \{s\}$ and e is the inclusion. Let $\varphi(x) = \varrho_Y(x, s)$. Then φ is a Katětov function on X and hence, setting $g(s) = \varphi$ and $g(x) = \widehat{x}$ for $x \in X$, we obtained the required embedding. \square

Exactly the same arguments show the existence of a Katětov functor for finite metric spaces with rational distances, leading to the rational Urysohn space. Finally, one can restrict the set of distances to the unit interval $[0, 1]$ obtaining a Katětov functor leading to the Urysohn sphere (or its rational variant). On the other hand, knowing that the category of finite metric spaces has one-point extension pushouts, Theorem 2.15 provides another Katětov functor on the category of finite rational metric spaces. The original Katětov functor is better in the sense that, when working in the category of *all* finite metric spaces, its values are complete separable metric spaces, which can be viewed as “minimal” spaces realizing all one-point extensions.

5.1 Conclusion

As we have seen above, the original Katětov construction deals with complete metric spaces, therefore it is formally out of the scope of our model-theoretic approach. The same applies to the recent Ben Yaacov’s construction [2] of a Katětov functor on separable Banach spaces, leading to the so-called *Gurariĭ space*, the unique universal separable Banach space that is *almost homogeneous*, namely, isometries between finite-dimensional subspaces can be approximated by bijective isometries of the entire space. Both examples can be presented in the framework of continuous model theory [1]. In the definition of a Katětov functor one would need to relax the extension property, as the Gurariĭ space satisfies only its approximate variant.

With some effort, one can adapt most of our arguments to categories of continuous models, obtaining in particular the universality result of Uspenskij [20] as well as its counterpart concerning monoids of nonexpansive mappings. We have decided to present the theory of Katětov functors in discrete model-theoretic setting in order to make it more clear and accessible.

It is possible to provide a purely category-theoretic framework for Katětov functors. Another direction is to study uncountable iterations of Katětov functors, obtaining models of arbitrary cardinality that are homogeneous with respect to their finitely generated substructures. This will be done elsewhere.

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